Math 2406 Final Exam, 2009

April 30, 2009

1. Define the following terms.
  a. Orthonormal basis.
  b. Minimal polynomial for a linear transformation.
  c. Multilinear mapping.
  d. Axioms of an inner product.
  e. Eigenspace.

2. Suppose that A and B are subspaces of a finite-dimensional vector space V. Prove that

$$\dim(A) + \dim(B) \;=\; \dim(A + B).$$

if and only if

$$A \cap B \;=\; \{0\}$$

(that is, A and B have only the 0 element in common.)

3. Suppose that $T : \mathbb{C}^n \to \mathbb{C}^n$, $n \geq 2$, is a linear transformation. Show that the linear transformations $1, T, T^2, T^3, ...$ do *not* span $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$. (The mapping 1 is the identity map – i.e. $1(x) = x$; and, the mappings T^j denotes the j-fold composition of T with itself. Hint: Can you give an upper bound on the dimension of the space spanned by these powers of T?)

4. Show that the set of $n \times n$ Hermitian matrices forms a subspace of all $n \times n$ matrices, where we take the field of scalars (coefficient space) to be $\mathbb{R}$ (even though the entries of these matrices are in $\mathbb{C}$). If the field of scalars (coefficients) is $\mathbb{C}$ instead, would we still get that Hermitian matrices form a subspace?

5. Suppose that $a_1, a_2, ..., a_n$ are non-zero real numbers. Let

$$A \ := \ \begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & 0 & 0 & \cdots & 0 & b_2 \\ 0 & 0 & a_3 & 0 & \cdots & 0 & b_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ c_1 & c_2 & c_3 & c_4 & \cdots & c_{n-1} & a_n \end{bmatrix},$$

where $a_n = b_n = c_n$. Write down a compact expression for the determinant and prove your answer (don't give me some baloney like one of the formulas you know for the determinant – find the determinant in a better way!).

6. Find an orthogonal basis that spans the same space as the vectors

$$(1, 2, 3, 4),\ (1, -1, 0, 0),\ (1, 2, 0, -7).$$

7. Consider the following system of equations

$$\begin{aligned} 4x + 2y + z + w &= 3 \\ 2x + w &= 4 \\ 8x + 4y + 2z + 3w &= 9 \end{aligned}$$

Express the solution set to this system as

$$v_0 + t_1 v_1 + t_2 v_2 + \cdots + t_k v_k,$$

where v_0 is a particular solution and $t_1, t_2, ...$ are free variables, and $v_1, v_2, ...$ belong to the kernel of a certain matrix.

8. Consider the matrix

$$A \ := \ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find matrices M and Λ, where Λ is diagonal, such that

$$A \ = \ M \Lambda M^{-1}.$$

9. Compute the determinant of the following matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & -3 & 4 \end{bmatrix}.$$

10. The following matrix consists of a single 3×3 Jordan block:

$$\begin{bmatrix} 0 & -3 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Thinking of this matrix as a transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$, find a basis B for $\mathbb{C}^3$ so that the matrix for T with respect to B (B is used for both the source and destination space) has the form

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$