Math 2406 Final Exam, 2009

April 29, 2009

1. Define the following terms.

a. Geometric multiplicity (in the context of eigenvalues and eigenvectors).

b. Linear Transformation.

c. Cofactor.

d. $\mathcal{L}(V, W)$.

e. Projection of a vector x onto a subspace V of a vector space W (explain how it is computed).

2. An $n \times n$ matrix with non-negative real entries is said to be "doubly stochastic" if the sum of entries in each row, and the sum of entries in each column, all equal 1. Prove that if A and B are both doubly stochastic, then AB is as well. (Hint: One way to solve it is to think in terms of eigenvectors and transposes.)

3. Suppose that V is the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous. Let W be the set of these that vanish at $x = 1$ (i.e. $f(1) = 0$). Prove that W is a subspace of V.

4. Suppose that $f_1, f_2, ..., f_{n^2+1}$ are linear transformations mapping $\mathbb{C}^n \to \mathbb{C}^n$. Show that there exist scalars $c_1, ..., c_{n^2+1}$, not all 0, such that

$$c_1 f_1 + c_2(f_1 \circ f_2) + c_3(f_1 \circ f_2 \circ f_3) + \cdots + c_{n^2+1}(f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_{n^2+1})$$

is the 0 transformation.

5. Suppose that A is an $m \times n$ matrix with real entries, $m < n$, having rank m. Show that the square matrix AA' (A' means the transpose of A) has rank m as well, by the following steps.

a. First, show that if the rows of A are mutually orthogonal, then the conclusion holds.

b. Next, show that even if the rows of A are not orthogonal, then you can find an invertible $m \times m$ matrix G such that GA *does* have orthogonal rows (what is the only way you know to orthogonalize a basis?).

c. Finally, consider $(GA)(GA)'$ and properties of determinants. Caution!: you cannot take the determinant of A because A is not a square matrix; but, there are other matrices in this problem that you can apply the determinant to.

6. Determine the matrix of the linear transformation $f(x, y, z) = (2x + 3y, z, x - y)$ with respect to the standard basis. Then, find the matrix of f with respect to the basis

$$(1, 1, 0),\ (0, 1, 1),\ (1, 0, 1).$$

7. The following matrix

$$\begin{bmatrix} 7/2 & 1/2 & -1/2 & 0 \\ 0 & 3 & 0 & 0 \\ 1/2 & 1/2 & 5/2 & 0 \\ 0 & -2 & 2 & 3 \end{bmatrix}$$

has only the eigenvalue $\lambda = 3$. Determine the number and sizes of all the Jordan blocks of this matrix when it is put into Jordan canonical form.

8. Find an orthogonal basis for the vector space spanned by $(1, 2, 4, 0), (0, 1, 1, 0)$, and $(0, 3, 1, 4)$ using Gram-Schmidt.

9. Find all solutions to the following system

$$\begin{aligned} x + 2y - w &= 1 \\ -3x - 6y + 5z - 7w &= 1 \\ x + 2y - 4z + 7w &= 1. \end{aligned}$$

Write your answer as

$$v_0 + t_1 v_1 + t_2 v_2 + \cdots,$$

where v_0 is a particular solution vector, and where $t_1, t_2, \ldots$ are free variables.

10. Find the eigenvalues and eigenvectors of the following matrix

$$\begin{bmatrix} 1 & -8 & 6 \\ 3 & -13 & 9 \\ 3 & -14 & 10 \end{bmatrix}.$$