Study sheet for Midterm 2, Math 2406, Spring 2009

April 6, 2009

- First, you should know all the material up to and including the first midterm.
- Know what a linear transformation is, and know how to prove or disprove that various maps $T : V \to W$ are (or are not) linear transformations.
- Know the definition of nullspace. Know the fact that if $T : V \to W$, and V has dimension n, then

  $$\text{rank}(T) + \text{nullity}(T) \;=\; n.$$

  It might also be a good idea to have some basic understanding of how this is proved.
- Know what it means for a mapping $f : V \to W$ to be surjective, injective, bijective. If, in addition f is a linear transformation, know the fact that f injective if and only if the kernel of f is trivial if and only if the nullity of f is trivial. What does knowing that f surjective tell you about the rank of f? What does knowing that f bijective tell you about the relationship between the dimension of V and the dimension of W?
- Know the definition of $L(V, W)$ – it is the set of linear transformations from $V \to W$. Know that this set of linear trasformations is itself a vector space, where $+$ is defined as follows: if f, g are linear transformations, then the sum $h = f + g$ is the map that sends

  $$x \;\to\; h(x) = (f+g)(x) = f(x) + g(x).$$

This map h is also a linear transformation. Scalar multiplication is defined in the obvious way. There is one additional operation that you can do on linear transformations, and that is compose them: if $f : V \to W$ and $g : W \to X$, are linear transformations, then the composition $h = gf = g \circ f$, which maps

$$x \ \to \ h(x) = (gf)(x) = g(f(x))$$

is a linear transformation from $V \to X$.

- Know that compositions of maps is associative.
- Know the definition of left and right inverse. Know the relationship between left and right inverse and injective and surjective maps (as I did it in class, not how the book did it). Know that if f has both a left and right inverse, then they must be the same, and know the proof of this fact (it makes use of associtivity of compositions of maps).
- Know that a linear transformation $f : V \to W$ is completely determined by where it sends a basis for V. Know how to write down the matrix for f with respect to bases B of V and C of W. Know how to convert such a matrix into another matrix, such that elementary bases are used for V and W. Know how to go the other way as well: start with the matrix using elementary bases, and then produce the matrix with respect to arbitrary bases B and C. There is a fairly straightforward way to do this using "change of basis matrices", which we have not talked about yet – we will discuss this when we talk about "similar transformations".
- Know that the composition of two transformations corresponds to the product of the corresponding matrices, when compatible bases are used (and it really is important that **compatible bases** be used).
- It is a very good idea to get some practice with matrices for f using different bases, because I will give you a question about it on your exam.
- Know the "isomorphism theorem" for matrices and linear transformations. It says that $L(V, W)$ is isomorphic to the set of $m \times n$ matrices, where m is the dimension of W and n is the dimension of V. Basically,

this theorem just says that, after fixing bases for V and W, we can match up each $f \in L(V, W)$ with a corresponding matrix, denoted by $m(f)$, such that f's action on V is structurally the same as $m(f)$'s action on V.

- Practice using the formulae for finding the i, j entry in the product AB of two matrices. Be able to prove basic things using these formulae. For example, be able to show that $(AB)' = B'A'$, where C' denotes the transpose of C.

- Know how to prove that the matrix product operation is associative, using the connection with maps. Basically, since the composition of maps is associative, and since matrix products can be thought of as compositions of maps (i.e. $m(fg) = m(f)m(g)$), it follows that matrix multiplication is also associative.

- Know how to do basic Gaussian elimination to solve systems of equations. Know how to spot free variables and bound variables. Know how to determine the kernel of a map using Gaussian elimination. Know how to find the inverse of a matrix.

- Know some basic facts about the determinant, such as that it satisfies the three axioms: (1) multiply a row by a scalar t, and the determinant changes by a factor of t; (2) add one row to another, and the determinant remains unchanged; and, (3) the identity matrix has determinant 1. Know how to prove basic facts using these axioms, such as: if a matrix has a 0 row, the determiant is 0; if a matrix has two rows the same, its determinant is 0; more generally, if the rows are dependent, then the determinant is 0; know how to show that if you interchange two rows, the determinant gets multiplited by -1.

- Know that the determinant of a diagonal matrix is the product of the diagonal entries. Know how to compute the determinant using Gaussian elimination (first, get the matrix into diagonal form).

- Know the fact that determinants are multilinear. Be able to apply this to prove basic things about the determiant of certain matrices. For example, be able to use this to prove that the determinant of a 2×2 matrix is $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

- Know the product formula for determinants – that the determinant of AB equals the determinant of A times the determinant of B. Using this, be able to prove that the determinant of the inverse of a matrix is the reciprocal of the determinant of that matrix. And, be able to show that an $n \times n$ matrix is singular if and only if its determinant is 0 (there are other ways to prove this as well, such as just using Gaussian elimination).

- Know the definition of the cofactor of a matrix, and know how to find the cofactor expansion of the determinant of a matrix. Know the relation between the inverse of a matrix and the cofactor matrix. Know how to find the determinant by expanding using minors.

- An interesting application of the cofactor formula for the inverse of a matrix is the following theorem: suppose that A is an $n \times n$ matrix with integer entries. Then, the entries of A^{-1} are all integers if and only if $\det(A) = \pm 1$.

  The proof of this goes as follows: if the determinant of A is ± 1, we see that $A(\mathrm{cof}(A))^t = \pm I_n$, and therefore $\pm \mathrm{cof}(A)$, which has all integer entries, is the inverse of A. Conversely, if A and A^{-1} have integer entries, their determinants are therefore integers, such that $\det(A) = 1/\det(A^{-1})$. The only integers such that their reciprocals are also integers, are ± 1. This then proves that $\det(A) = \pm 1$.

- Know the expansion for the determinant given by

  $$\sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

  Know how to use this to prove that a matrix and its transpose have the same determinant. Basically, this amounts to realizing that every term in this expansion involving $\sigma \in S_n$, can be paired up with an analogous term for when the determinant of the transpose is computed. The key fact that one uses is that

  $$\mathrm{sign}(\sigma) \;=\; \mathrm{sign}(\sigma^{-1}).$$

  The reason for this is that if

  $$\sigma \;=\; \tau_1 \circ \cdots \circ \tau_k,$$

where the τ_i are transpositions, then

$$\sigma^{-1} \;=\; \tau_k \circ \cdots \circ \tau_1.$$

- Recall basic facts about the sign of a permutation σ as described above. Basically, every permutation (bijection)

$$\sigma \;:\; \{1, 2, ..., n\} \;\rightarrow\; \{1, 2, ..., n\}$$

can be written as a product (composition) of a finite number of transpositions. And, no matter what set of these transpositions are used, the parity of k (the number used) is always the same. If k is even, σ is said to be an "even permutation" and the sign of σ is $+1$; but, if k is odd, then σ is said to be an "odd permutation", and its sign is -1.