Practice Math 110 Final

Instructions: Work all of problems 1 through 5, and work any 5 of problems 10 through 16.

1. Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 3 & 2 \\ 6 & 6 & 5 \end{bmatrix}.$$

a. Use Gauss elimination to reduce A to an upper triangular matrix (row reduced form).

b. Define what is meant by an elementary matrix.

c. Write down the elementary matrices corresponding to the row operations (interchange two rows, multiply a row by a scalar, and add one row to another).

2.

a. Define what is meant by the inverse of a matrix.

b. Find the inverse of the matrix A in question 1.

3.

a. Define what is meant by a symmetric matrix.

b. Show that if A and B are $n \times n$ matrices, and A is symmetric, then A^2 and $B'AB$ are symmetric (B' means the transpose of B).

c. Define what is meant by a permutation matrix P. Show that there exists a power $n \geq 1$ such that if P is a permutation matrix, then $P^n = I$, where I is the identity matrix.

4.

a. Let V be a vector space. Define what is meant by a basis for V. Also define the dimension of V.

b. Let A be an $m \times n$ matrix, and let U be the normal form of A. Define the row space of A, the column space of A, the rank of A, the nullity of A, and the nullspace of A.

5.

a. State the dimension theorem.

b. Given the linear map

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^4,$$

which sends

$$(a, b, c) \;\rightarrow\; (a + b, 2b, 3a + b, c + b),$$

find the matrix representation for T with respect to the bases

$$(1, 0, 0),\ (0, 1, 1),\ (1, 0, 1)$$

for $\mathbb{R}^3$, and

$$(1, 0, 0, 0),\ (0, 1, 1, 0),\ (0, 0, 1, 0),\ (1, 0, 0, 1)$$

for $\mathbb{R}^4$.

6. Suppose that A is an $n \times n$ matrix, and that v is an eigenvector of A with eigenvalue λ. If $P(x)$ is any polynomial, show that

$$P(A)v \;=\; P(\lambda)v.$$

Note: The $P(A)$ is a matrix, whereas the $P(\lambda)$ is a scalar.

7. Assume the claim in problem 6 is true, suppose that A is an $n \times n$ matrix, and that $f(x)$ is its minimal polynomial, and that $c(x)$ is its characteristic polynomial. Prove that if λ is a root of $c(x)$, then it is also a root of $f(x)$.

Hint: If λ is root of $c(x)$, then it is an eigenvalue, which must have a non-zero eigenvector v. Since $f(x)$ is the minimal polynomial, we must have $f(A) = 0$ (that is, $f(A)$ is the 0 matrix), and so

$$f(A)v = 0.$$

What can you conclude, assuming problem 6.

8. Let A be an $n \times n$ matrix with real entries. Prove that the subspace

$$\mathrm{Span}(I_n, A, A^2, A^3, ...)$$

has dimension $\leq n$ (even though the matrix A has n^2 entries and lies in the vector space $\mathrm{Mat}_{n\times n}(\mathbb{R})$, which has dimension n^2).

9. Suppose that V is a vector space equipped with an inner product $< \cdot, \cdot >$. Let v, w be two linearly independent vectors in V. Prove that the vectors

$$v \ \text{ and } \ w - \lambda v, \ \text{ where } \ \lambda = \frac{< v, w >}{< v, v >}$$

are orthogonal. You are NOT allowed to say that this follows directly from Gram-Schmidt. The point of the problem here is to show that you know some things about inner products, as well as the definition of orthogonal.

10.

a. Prove that a matrix and its transpose have the same eigenvalues.

b. Suppose P is a transition matrix of a markov chain, whose entries $P_{i,j}$ give the probability that a "atom" transitions from state j to state i, and note that the sum of the entries in each column is 1. Prove that $\lambda = 1$ is an eigenvalue for A (Hint: Use part a.).

11. Let N be the set of all nilpotent matrices in $\text{Mat}_{n\times n}(\mathbb{R})$. For $n \geq 2$, N is NOT a subspace; for example,

$$A \;=\; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \;\text{ and }\; B \;=\; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are both nilpotent, but their sum $A + B$ is not nilpotent, so this proves that N cannot be a subapce.

However, if W is any subspace of $\text{Mat}_{n\times n}(\mathbb{R})$ such that every member of W commutes with every other member of W under matrix multiplication, then prove that

$$W \,\cap\, N \quad \text{is a subspace of } \text{Mat}_{n\times n}(\mathbb{R}).$$

Hint: This problem is similar to the problem on your first exam, where you were asked to show that A, B nilpotent implies $A + B$ nilpotent.

12. Suppose that X, Y, Z are finite-dimensional vector spaces, and that $f : X \to Y$ and $g : Y \to Z$ are both linear maps. Prove that

$$\dim(\text{im}(g \circ f)) \;\leq\; \dim(\text{im}(f)).$$

Hint: In the composition $g \circ f$, the map f first maps X into $\text{im}(f)$, and then g maps $\text{im}(f)$ into $\text{im}(g)$. So this last map is actually $g|_{\text{im}(f)}$ (that is, g restricted to the image of f). Applying the dimension theorem to the map $g|_{\text{im}(f)}$ and the departure space $\text{im}(f)$, we get

$$\dim(\text{im}(f)) \;=\; \dim(\ker(g|_{\text{im}(f)}) + \dim(\text{im}(g|_{\text{im}(f)}).$$

Note that $\text{im}(g|_{\text{im}(f)}) = \text{im}(g \circ f)$. So,...

13. Suppose that V is a finite-dimensional vector space, and that T is a linear map from V to V. Further, let x be a vector in V, and let k be the least integer such that

$$x,\ T(x),\ ...,\ T^{k-1}(x) \ \text{ are all linearly independent.}$$

Then, show that the subspace

$$S \ = \ \text{Span}(x, T(x), ..., T^{k-1}(x))$$

is T-invariant; that is, show that if $s \in S$, then $T(s) \in S$.

14.

a. Define what is meant by a determinental map.

b. Show that if A is an $n \times n$ matrix with real entries, and D is a determinental map from $\text{Mat}_{n\times n}(\mathbb{R})$ to $\mathbb{R}$, then if A has two identical columns, $D(A) = 0$.

15. Label the following as true or false:

a. The rank of a matrix is equal to the number of its non-zero columns.

b. Elementary row operations applied to a matrix preserve the rank of a matrix.

c. If A is an $m \times n$ matrix, where $m \leq n$, then the rank of A is at least equal to m.

d. Any system of n linear equations in n unknowns has at least one solution.

e. Any system of n linear equations in n unknowns has at most one solution.

f. Any polynomial of degree n and leading coefficient $(-1)^n$ is the characteristic polynomial of some $n \times n$ matrix A.

g. The characteristic polynomial of a matrix always has degree larger than the minimial polynomial.

16. Assume A is an invertible $n \times n$ matrix with integer entries. Prove that the entries of A^{-1} have integer entries if and only if the $\det(A) = \pm 1$.

Hint: Look at the adjugate of A, and recall that

$$[\text{adj } A]_{i,j} \ = \ (-1)^{i+j}\det(A_{j,i}),$$

where $A_{j,i}$ is gotten by taking A and removing the jth row and ith column. Also recall that

$$A(\text{adj } A) \;=\; \det(A) I_n.$$

How can you use this to prove the claim that

$$A^{-1} \text{ has integer entries} \quad \Longrightarrow \quad \det(A) = \pm 1 \text{ ?}$$