Sample questions for Math 2406, midterm 1

February 18, 2009

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that maps each subspace of dimension 2 one-to-one to some other subspace of dimension 2. Show that the only vector v for which $f(v) = 0$ is $v = 0$.

2. Suppose that X is a subspace of a vector space V. Prove that the orthogonal complement of the orthogonal complement of X is X.

3. Prove that the cross product is non-associative, as well as non-commutative.

4. Fix a vector $x \in \mathbb{R}^3$. Consider the set of all pairs P of vectors

$$(v, w) \ \in \ \mathbb{R}^3 \times \mathbb{R}^3$$

satisfying

$$v \times (x \times w) \ = \ 0.$$

Is P a subspace of $\mathbb{R}^3 \times \mathbb{R}^3$?

5. Suppose that P_1 and P_2 are two planes in $\mathbb{R}^3$, and let L be a line in $\mathbb{R}^3$ that hits both P_1 and P_2. Suppose that

$$|L \cap P_1| \ + \ |L \cap P_2| \ \geq \ 3.$$

Prove that $P_1 \cap P_2$ contains a line.

6. For which triples of vectors u, v, w is the scalar triple product $[u, v, w] > 0$? (Describe the triples.)

7. Suppose that A, B, C are elements of some vector space V. Further, suppose that

$$||A + B + C||^2 \ = \ ||A||^2 + ||B||^2 + ||C||^2.$$

Is it the case that A, B, C are all mutually orthogonal? What about if you just use two vectors A, B (i.e. does $||A + B||^2 = ||A||^2 + ||B||^2$ imply A and B orthogonal?)?

8. What is the difference between normal and orthogonal?

9. Suppose that X is a subspace of a finite dimensional vector space V.
a. Prove that X is also finite dimensional.
b. Prove that $X^{\perp}$ is finite dimensional.

10. Fix a vector space V. Do the set of subspaces of V form a vector space? Here, addition of subspaces is defined as follows: Given subsapces A, B, we have

$$A + B \ = \ \{a + b \ : \ a \in A, \ b \in B\}.$$

And, scalar mutiplication is

$$\lambda A \ := \ \{\lambda a \ : \ a \in A\}.$$