

Bijection and Cardinality

Discrete Mathematics
Slides by Andrei Bulatov

Previous Lecture

- Functions
- Describing functions
- Injective functions
- Surjective functions
- Bijective functions

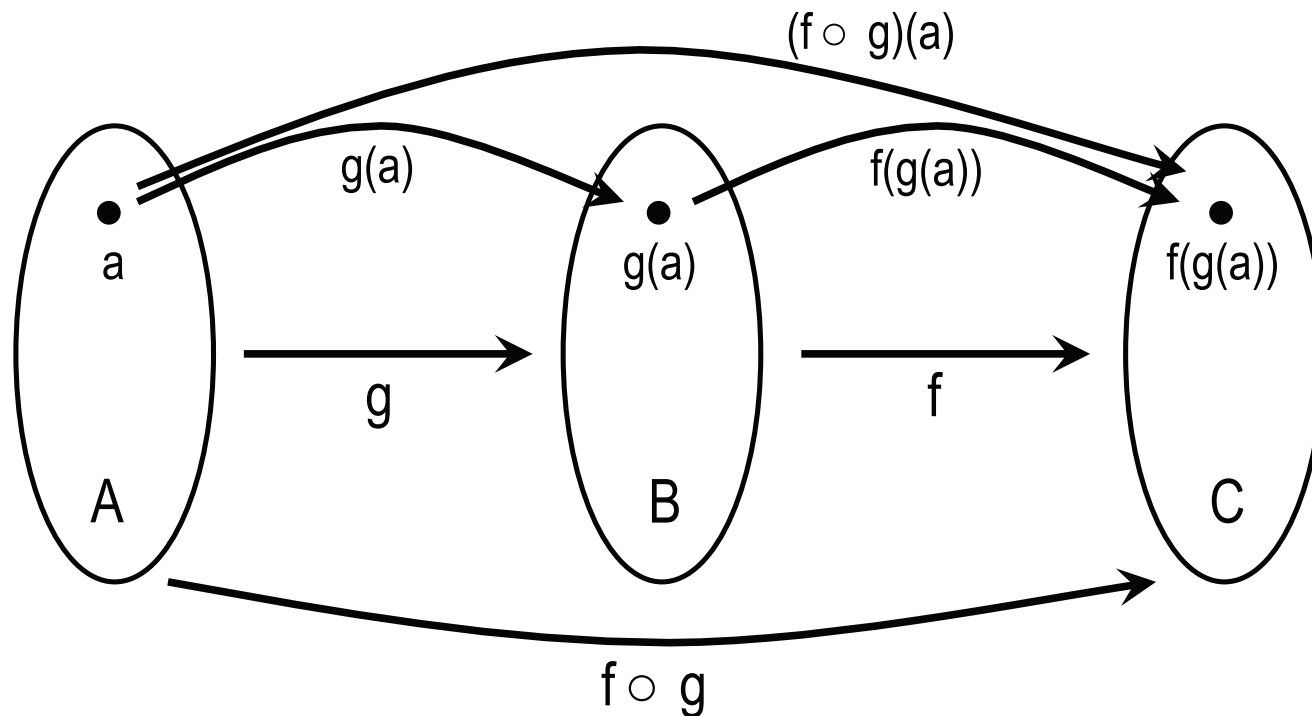
Properties of Functions

- A function f is said to be **one-to-one**, or **injective**, if and only if $f(a) = f(b)$ implies $a = b$.
- A function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a **surjection** if it is onto.
- A function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto.

Composition of Functions

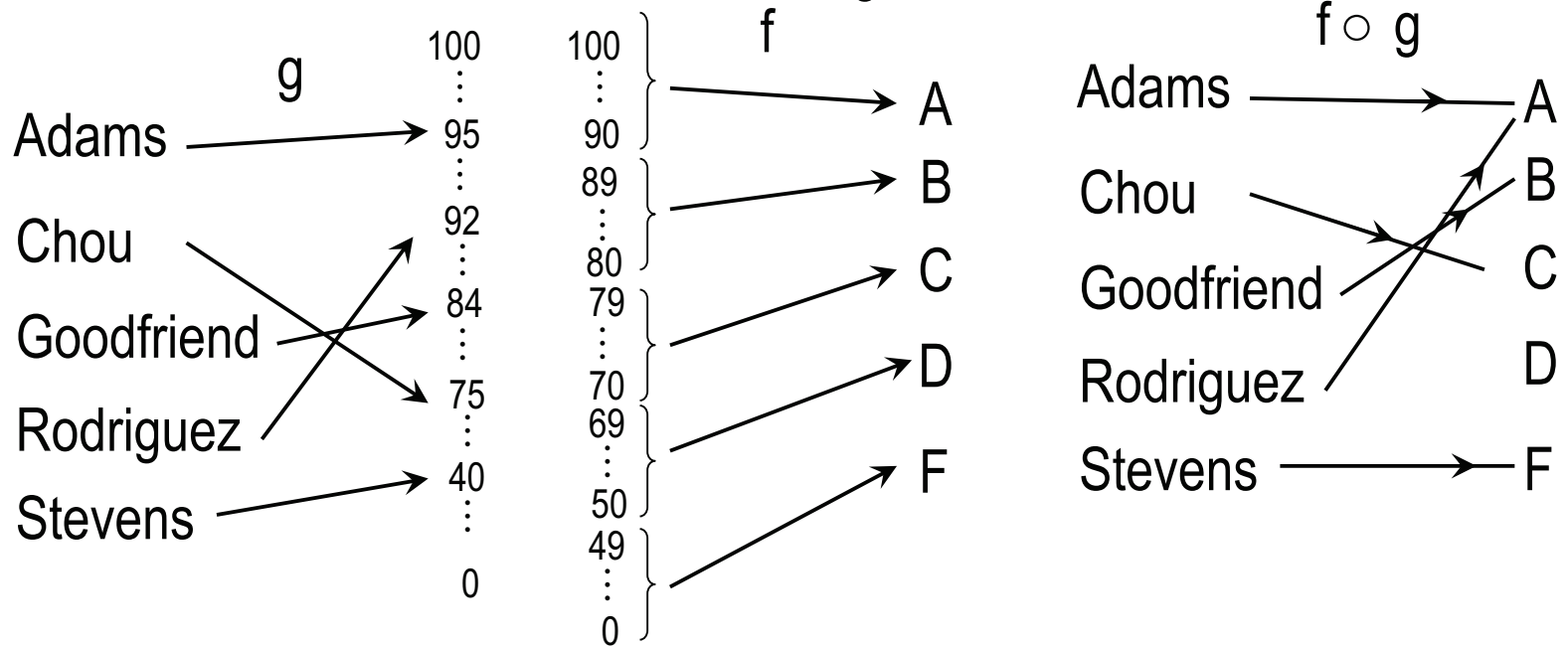
- Let g be a function from A to B and let f be a function from B to C . The **composition** of the functions f and g , denoted by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a))$$



Composition of Functions (cntd)

- Suppose that the students first get numerical grades from 0 to 100 that are later converted into letter grade.



- Let $f(a) = b$ mean 'b is the father of a'.
What is $f \circ f$?

Composition of Numerical Functions

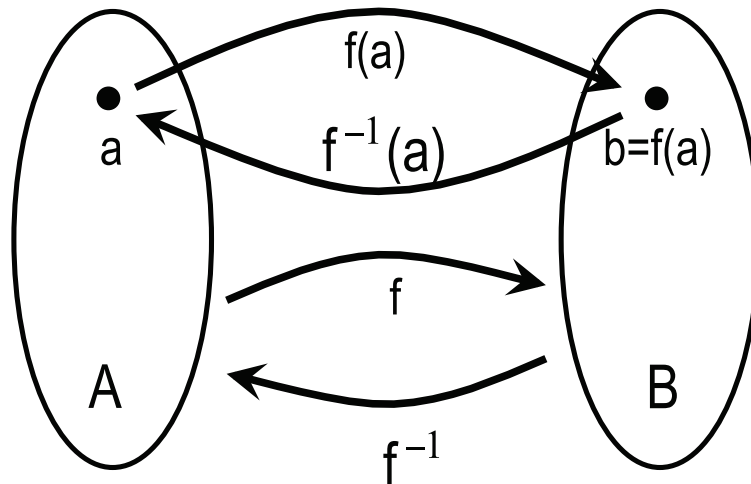
- Let $g(x) = x^2$ and $f(x) = x + 1$. Then
$$(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1$$
- Thus, to find the composition of numerical functions f and g given by formulas we have to substitute $g(x)$ instead of x in $f(x)$.

Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B . The **inverse function** of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

The inverse function is denoted by f^{-1} .

Thus, $f^{-1}(b) = a$ if and only if $f(a) = b$



Note!

f^{-1} does not mean $\frac{1}{f(x)}$

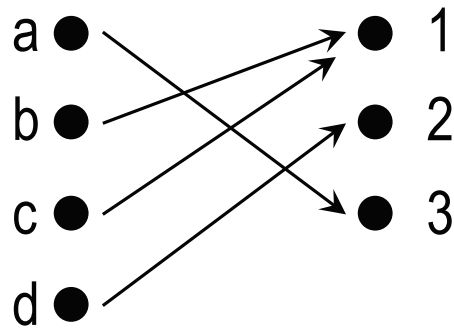
$$f \circ f^{-1} = i_B$$

$$f^{-1} \circ f = i_A$$

Inverse Functions (cntd)

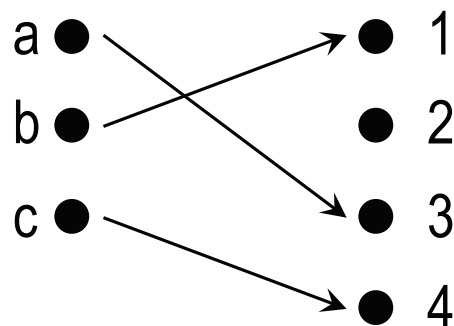
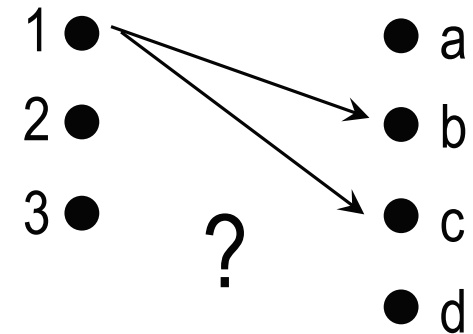
- If a function f is not a bijection, the inverse function does not exist.
Why?

- If f is not a bijection, it is either not one-to-one, or not onto



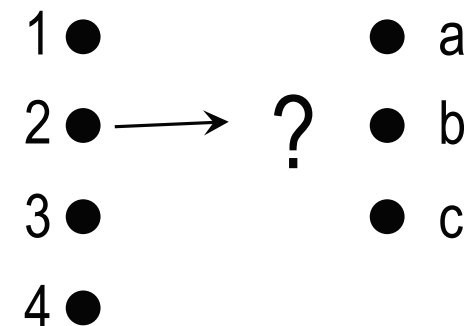
onto, not one-to-one

$$f^{-1}(1) = ?$$



one-to-one, not onto

$$f^{-1}(2) = ?$$



How to Count Elements in a Set

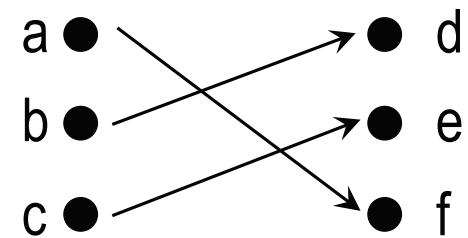
- How many elements are in a set?
- Easy for finite sets, just count the elements.
- What about infinite sets? Does it make sense at all to ask about the number of elements in an infinite set?
- Can we say that this infinite set is larger than that infinite set?
- Which set is larger: the set of all integers or the set of even integers?
 - the set of all integers or the set of all rationals?
 - the set of all integers or the set of all reals?

Cardinality and Bijections

- If A and B are finite sets, it is not hard to see that they have the same cardinality if and only if there is a bijection from A to B
- For example, $|\{a,b,c\}| = |\{d,e,f\}|$

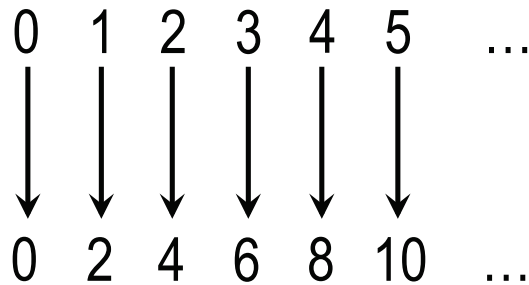
$\{ a \ b \ c \}$

$\{ d \ e \ f \}$



Cardinality and Bijections

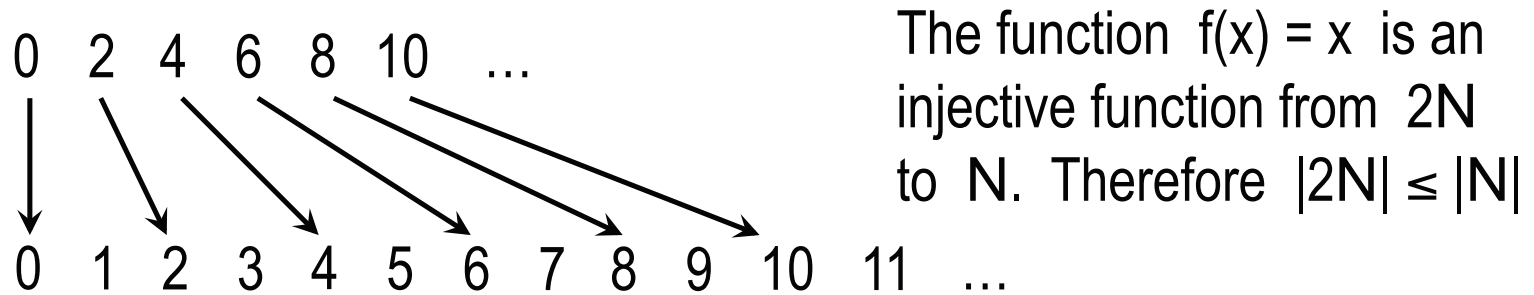
- Sets A and B (finite or infinite) have the same cardinality if and only if there is a bijection from A to B
- $|\mathbb{N}| = |2\mathbb{N}|$



The function $f: \mathbb{N} \rightarrow 2\mathbb{N}$, where
 $f(x) = 2x$,
 is a bijection

Comparing Cardinalities

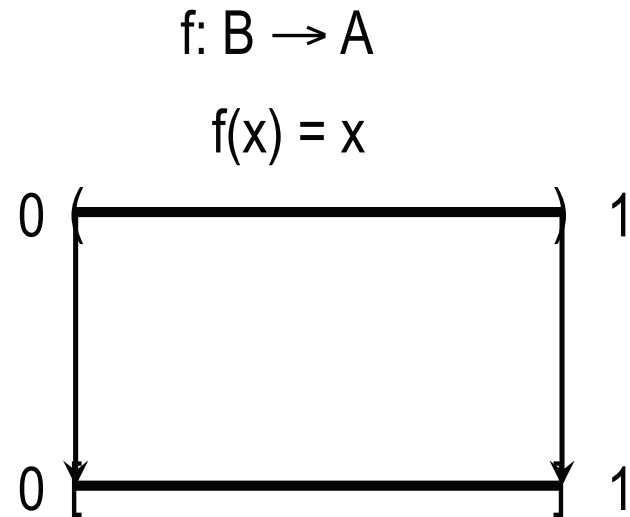
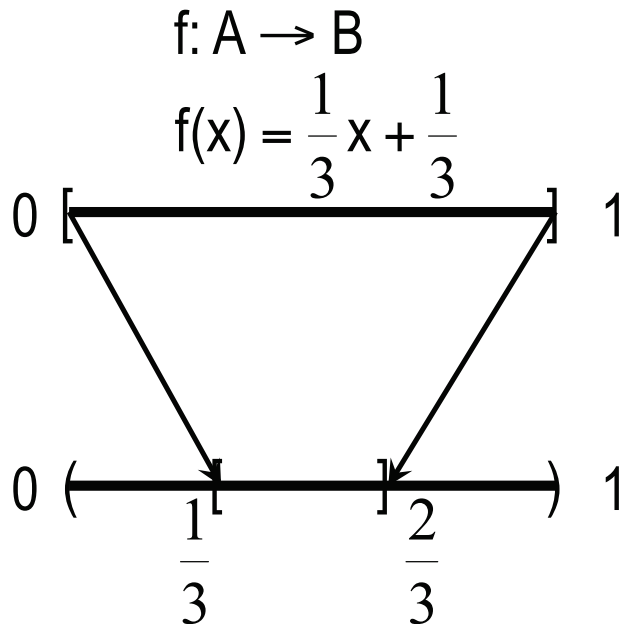
- Let A and B be sets. We say that $|A| \leq |B|$ if there is an injective function from A to B .



- If there is an injective function from A to B , but not from B to A , we say that $|A| < |B|$
- If there is an injective function from A to B and an injective function from B to A , then we say that A and B have the same cardinality
- Exercise: Prove that a bijection from A to B exists if and only if there are injective functions from A to B and from B to A .

Example

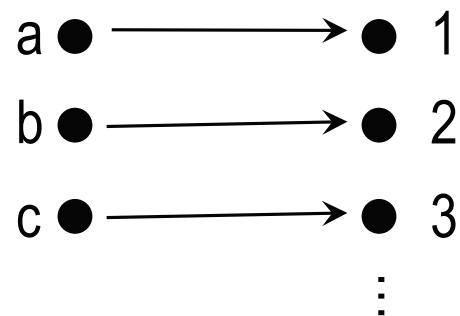
- Let A be the closed interval $[0;1]$ (it includes the endpoints) and B – the open interval $(0;1)$ (it does not include the endpoints)
- There are injective functions f and g from A to B and B to A , respectively.



Countable and Uncountable

- A set A is said to be **countable** if $|A| \leq |\mathbb{N}|$
- This is because an injective function from A to \mathbb{N} can be viewed as assigning numbers to the elements of A , thus counting them
- Sets that are not countable are called **uncountable**
- Countable sets:

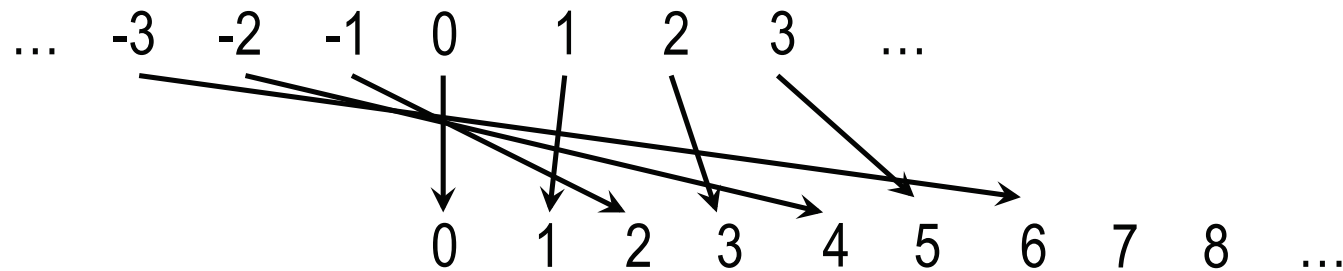
finite sets



any subset of \mathbb{N}

More Countable Sets

- The set of all integers is countable



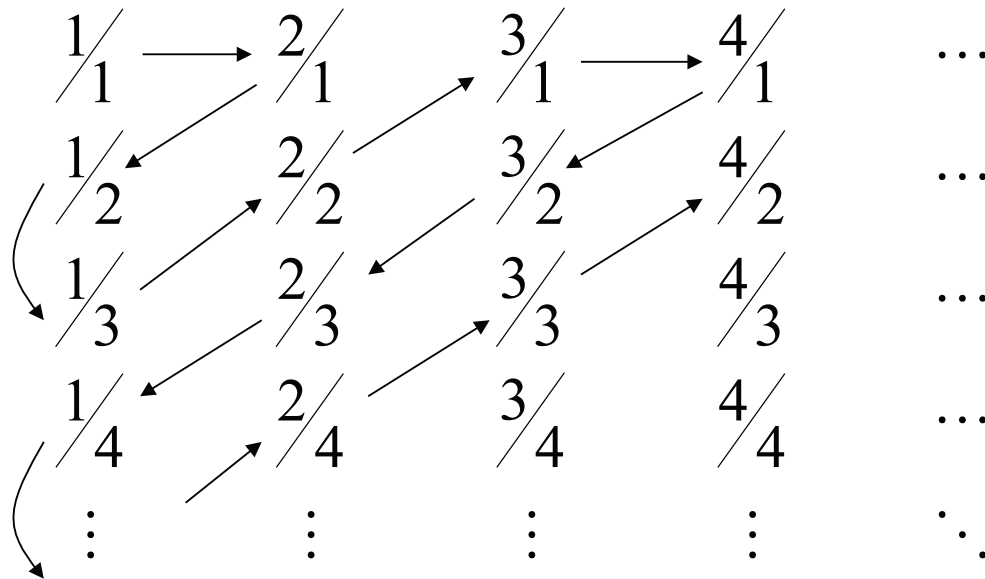
- In other words we can make a list of all integers

$0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \dots$

- The cardinality of the set of all natural numbers is denoted by \aleph_0

More Countable Sets (cntd)

- The set of positive rational numbers is countable
- Every rational number can be represented as a fraction $\frac{p}{q}$
We do not insist that p and q do not have a common divisor



- This gives an injection from \mathbf{Q}^+ to \mathbf{N} . The converse injection is $f(x) = x + 1$

The Smallest Infinite Set

- Theorem.

If A is an infinite set, then $|A| \geq \aleph_0$

- Proof requires mathematical induction. Wait for a week.

Uncountable Sets

- Can we make a list of all real numbers?
- Every real number can be represented as an infinite decimal fraction, like $3.1415926535897932384626433832795028841971\dots$
- Suppose we have constructed a list of all real numbers

$$1. \quad a_{10}.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}\dots$$

$$2. \quad a_{20}.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}a_{27}\dots$$

$$3. \quad a_{30}.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36}a_{37}\dots$$

$$4. \quad a_{40}.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46}a_{47}\dots$$

$$5. \quad a_{50}.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56}a_{57}\dots$$

$$\vdots \quad \quad \quad \vdots$$

Here the a_{ij} are
digits $0, 1, 2, \dots, 9$

Let

$$b_i = \begin{cases} 4, & \text{if } a_{ii} \neq 4, \\ 5 & \text{otherwise} \end{cases}$$

- It is not hard to see that the number $0.b_1b_2b_3b_4b_5b_6b_7\dots$ is not in this list

Cantor's Theorem

● Theorem (Cantor). For any set $|P(A)| > |A|$.

Proof.

Suppose that there is a bijection $f: A \rightarrow P(A)$.

We find a set that does not belong to the range of f . A contradiction with the assumption that f is bijective.

Consider the set $T = \{ a \in A \mid a \notin f(a) \}$

If T is in the range of f , then there is $t \in A$ such that $f(t) = T$.

Either $t \in T$ or $t \notin T$.

If $t \in T$ then $t \in f(t)$, and we get $t \notin T$.

If $t \notin T$ then $t \in T$.

Q.E.D.

Cantor's Theorem (cntd)

- This method is called Cantor's diagonalization method
- The cardinality of $P(A)$ is denoted by $2^{|A|}$
- Thus, we obtain an infinite series of infinite cardinals

$$|\mathbf{N}| = \aleph_0$$

$$2^{\aleph_0} = \aleph_1 \quad (= |\mathbf{R}|)$$

$$2^{\aleph_1} = \aleph_2$$

$$\vdots$$

Continuum Hypothesis

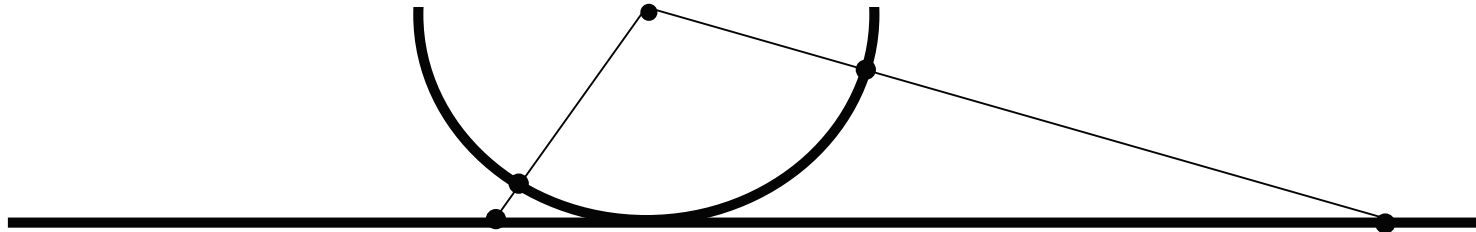
- We just proved that $\aleph_0 < |R|$. Does there exist a set A such that $\aleph_0 < |A| < |R|$?
- The negative answer to this question is known as the **continuum hypothesis**.
- Continuum hypothesis is the first problem in the list of Hilbert's problems
- Paul Cohen resolved the question in 1963. The answer is shocking: You can think either way.



More Uncountable Sets

- For any real numbers a, b , the open interval $(a;b)$ has the same cardinality as \mathbb{R}

a (—————) b



Homework

Exercises from the Book:

No. 1def, 2b, 4 (page A-32)

- Construct a bijective mapping between the closed interval $[0;1]$ and the square $[0;1] \times [0;1]$