Study sheet for Midterm 2, Math 2406, Spring 2012

April 21, 2012

- First, you should know all the material up to and including the first midterm.
- Know what is meant by a set of independent vectors; know what dependent means (in the context of vectors). Know that example about $u_i(x) = e^{a_i x}$ – these are independent functions in the vector space of all functions from $\mathbb{R} \to \mathbb{R}$ if the a_i's are all distinct; know the proof of this fact.
- Know the fact that if $x_1, ..., x_{k+1} \in L(S)$, where S is any set of k or fewer vectors, then those x_i's must be dependent. Know how to use this fact to prove that the dimension of a vector space is well-defined: the idea is that if $b_1, ..., b_k$ and $c_1, ..., c_\ell$ are bases, then we can't have $k < \ell$ or $k > \ell$ – the former is ruled-out by the fact that it would imply that $c_1, ..., c_\ell \in L(\{b_1, ..., b_k\})$, and therefore that $b_1, ..., b_k$ are dependent.
- Know the definition of basis and dimension. Know that if a space has two bases with finitely many elements, then those bases must have the same size (that size is the dimension). Know that any set of n independent vectors in a space of dimension n, is necessarily a basis.
- Know that any set of independent vectors can be extended to a basis. That is, if $\{x_1, ..., x_k\}$ are independent in a space of dimension n, then there exist $x_{k+1}, ..., x_n$ such that $\{x_1, ..., x_n\}$ is a basis.
- Know what is meant by an inner product. There are 4 axioms – know them. Also know what is meant by a complex inner product (in particular, know the Hermitian symmetry property). Know what is meant

by a "Euclidean space" (basically, it is a vector space equippend with inner product).

- Know the standard examples of inner products: the dot product; the integral inner product $(f, g) = \int_a^b f(x)g(x)dx$; know the weighted versions of these; etc.

- Know the general Cauchy-Schwarz inequality $|(f, g)|^2 \leq (f, f)(g, g)$.

- Know how to construct a norm $\|f\|$ from an inner product: you can just define $\|f\| = (f, f)^{1/2}$. A "norm" has the defining properties: $\|x\| > 0$ if $x \neq 0$; $\|x\| = 0$ if $x = 0$; $\|cx\| = |c|\|x\|$; and, the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$. Know how to prove the triangle inequality using the Cauchy-Schwarz inequality.

- Know what is meant by orthogonal vectors. There are basically two equivalent definitions you could use: definition 1 is that $x_1, ..., x_k$ are orthogonal if all pairwise inner-products (x_i, x_j), $i \neq j$, are 0; and definition 2 is that those vectors satisfy a kind of Pythagorean Theorem, in that $\|x_1 + \cdots + x_k\|^2 = \|x_1\|^2 + \cdots + \|x_k\|^2$.

- Know some examples of orthogonal vectors – like, for instance, the functions $e^{2\pi it}$, with respect to the inner product $(f, g) = \int_0^1 f(x)\overline{g(x)}dx$.

- Know that if $x_1, ..., x_n$ are orthogonal, then they must be linearly independent. Obviously, however, not every set of linearly independent vectors is orthogonal.

- Know the definition of orthonormal. Know that if $v_1, ..., v_n$ are orthonormal, then any vector x in the span of those vectors may be written as $x = \sum_i (x, v_i)v_i$. Know Parseval's formula: $(x, y) = \sum_i (x, v_i)\overline{(y, v_i)}$. Know the proofs of these facts.

- Know how the Gram-Schmidt process works to produce an orthogonal set of vectors spanning the same space as a given set of vectors. Be able to do examples of applying Gram-Schmidt.

- Know the definition of the projection of a vector x onto a subspace S. Know how to compute it, given S and the inner-product. Letting $p(x)$ denote this projection, know that $p(x)$ has the property that $\|p(x) - x\| \leq \|y - x\|$, for any $y \in S$; that is, $p(x)$ comes closer to x than any other vector in S. Know what this means in terms of Fourier series (see the book); know how to find the gram-schmidt basis for a given space of polynomials, given a particular inner-product.

- Know what a linear transformation is, and know how to prove or disprove that various maps $T : V \to W$ are (or are not) linear transformations.

- Know the definition of nullspace. Know the fact that if $T : V \to W$, and V has dimension n, then

  $$\text{rank}(T) + \text{nullity}(T) \; = \; n.$$

  It might also be a good idea to have some basic understanding of how this is proved.

- Know what it means for a mapping $f : V \to W$ to be surjective, injective, bijective. If, in addition f is a linear transformation, know the fact that f injective if and only if the kernel of f is trivial if and only if the nullity of f is 0. What does knowing that f surjective tell you about the rank of f? What does knowing that f bijective tell you about the relationship between the dimension of V and the dimension of W?

- Know the definition of $L(V, W)$ – it is the set of linear transformations from $V \to W$. Know that this set of linear trasformations is itself a vector space, where + is defined as follows: if f, g are linear transformations, then the sum $h = f + g$ is the map that sends

  $$x \; \to \; h(x) = (f + g)(x) = f(x) + g(x).$$

  This map h is also a linear transformation. Scalar multiplication is defined in the obvious way. There is one additional operation that you can do on linear transformations, and that is compose them: if $f : V \to W$ and $g : W \to X$, are linear transformations, then the composition $h = gf = g \circ f$, which maps

  $$x \; \to \; h(x) = (gf)(x) = g(f(x))$$

  is a linear transformation from $V \to X$.