Study Sheet for Math 2406 Midterm 1, Spring 2012

February 28, 2012

Your exam will cover all the material up to and including section 3.6 (subspaces) of your book.

The following items are not necessarily an exclusive list of the topics to be on your exam; they are intended only to help you study for the exam. (This statement is here so that people don't come to me and say, "But that wan't on the study sheet!" in the event I forget to list something.)

1. Know the basics of logic, and know what the symbols

   $$\vee \ \wedge \ \neg \ \Longrightarrow \longrightarrow \forall \, \exists \ \leftrightarrow \ \Longleftrightarrow$$

   mean. Know what a truth table is; know how to construct one; and know how to use it to prove a logical proposition. Know what "proof by contradiction" is; know what "proof by contraposition" is. Know that when you negate a statement, you change quantifiers around. Know de Morgan's laws.

2. Know what is meant by "type I" and "type II" induction. Know how to prove basic statements using induction, such as to show that $1+2+\cdots+n = n(n+1)/2$. Also, study the proof that every integer $n \geq 2$ can be written as a product of prime numbers (using type II induction). Recall the proof on the number of moves needed to solve the Towers of Hanoi puzzle. Know how to use induction to prove inequalities.

3. Know basic facts about sets, and in particular what the symbols $\cup \cap$ $\emptyset$ $\in \subseteq \subset \supseteq \supset$ and $\overline{A}$ all mean. Know what is meant by the "universal set" or "universe". Know what injection, surjection, and bijection all

mean; know that $|A| = 5$ means that there exists a bijection $\varphi : A \to \{1, 2, ..., 5\}$; and know that $|A| = |B|$ means there exists a bijection $\psi : A \to B$. Know what is meant by "countable" and "uncountable". Know that the real numbers $\mathbb{R}$ form an uncountable set; know that $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$ are all countable. Know the de Morgan laws of sets, and be able to prove them. Know that the statement "$A \subseteq B$" can be rendered in logical symbols as "$\forall x (x \in A \Longrightarrow x \in B)$". Know the de Morgan laws.

4. Know what is meant by an equivalence relation; and know what the terms symmetric, reflexive, and transitive mean. Know how to prove that certain relations are equilvalence relations – see the note on the course webpage about equivalence relations.

5. Know basic properties of the vector space $\mathbb{R}^n$. Know how to think about vectors as points in the space; know how to draw vectors; know how to add vectors; know what is meant by parallel vectors; know what is meant by "a scalar multiple of a vector". Know what is meant by the dot product $A \cdot B$, and know its basic proerpties; know the difference between the complex dot product and the real dot product; know what is meant by $\|A\|$, and how to express it in terms of dot product as $(A \cdot A)^{1/2}$; know the interpretation of the dot product in terms of projections; know what is meant by "orthogonal vectors". Know the Cauchy-Schwarz inequality for $\mathbb{R}^n$ and $\mathbb{C}^n$. Know the triangle inequality $\|A + B\| \leq \|A\| + \|B\|$.

6. Know the definition of a line in $\mathbb{R}^n$. Know how to find the point on a line that is closest to the origin, as well as how to derive the formula. Know basic facts of geometry derivable from this definition of a line; in particular, the parallel postulate. Know the definition of a plane, what it means for two planes to be parallel, and the various conditions guaranteeing when two planes are equal. Know how to derive the formula for the point on a plane closest to the origin. Know the cross-product in $\mathbb{R}^3$; know the "right hand rule"; and know how to use the cross-product to find a third vector, orthogonal to two given given vectors.

7. Know what is meant by "the linear span of the vectors $x_1, ..., x_k$", which, in symbols, is written $L(\{x_1, ..., x_k\})$ (not to be confused with $L(P; A)$, our notation for a line). Know what it means for a set of vectors in $\mathbb{R}^n$ to be "linearly dependent" (often written as just 'dependent') and

"linearly independent" (often written as just 'independent'). Know the fact that if $x_1, ..., x_{k+1} \in L(\{y_1, ..., y_k\})$, then $x_1, ..., x_{k+1}$ must be linearly dependent. Know that if $x_1, ..., x_k$ are "orthogonal" (meaning that the dot-products $x_i \cdot x_j = 0$ for all $i \neq j$; and that no $x_i = 0$), then they are linearly independent; and know how to prove this. Know the terms basis and dimension. Know basic properties, such as that any n linearly independent vectors in $\mathbb{R}^n$ necessarily form a basis; and that any two bases have the same number of elements (that number is the dimension).

8. Know how to prove that something is a vector space (relative to a given set of scalars), given a list of the axioms of a vector space (I won't expect you to be able to memorize the complete list; but will expect you to be able to apply the axioms); know how to prove the uniqueness of the 0-vector; know how to prove the uniqueness of additive inverses $-x$; know how to prove basic properties of a vector space (such as those in Theorem 3.3, page 95). Know some examples of a vector space (such as all function $f : \mathbb{R} \to \mathbb{R}$ vanishing at $x = 1$; all polynomials; all planes through the origin; etc.). Know what is meant by a subspace $S \subseteq V$ contained in a vector space V; and know how to prove that S is a subspace (just show it is non-empty and satisfies the closure axioms).