

Study Guide for Math 2406 Final, Fall 2012

December 10, 2012

Here are a few items that you should know for the final exam.

- Know the basics of logic, and know what the symbols

$$\vee \wedge \neg \implies \longrightarrow \forall \exists \leftrightarrow \Longleftrightarrow$$

mean. Know what a truth table is; know how to construct one; and know how to use it to prove a logical proposition. Know what “proof by contradiction” is; know what “proof by contraposition” is. Know that when you negate a statement, you change quantifiers around. Know de Morgan’s laws.

- Know what is meant by “type I” and “type II” induction. Know how to prove basic statements using induction, such as to show that $1 + 2 + \cdots + n = n(n + 1)/2$. Also, study the proof that every integer $n \geq 2$ can be written as a product of prime numbers (using type II induction). Recall the proof on the number of moves needed to solve the Towers of Hanoi puzzle. Know how to use induction to prove inequalities.

- Know basic facts about sets, and in particular what the symbols $\cup \cap \emptyset \in \subseteq \subset \supseteq \supset$ and \overline{A} all mean. Know what is meant by the “universal set” or “universe”. Know what injection, surjection, and bijection all mean; know that $|A| = 5$ means that there exists a bijection $\varphi : A \rightarrow \{1, 2, \dots, 5\}$; and know that $|A| = |B|$ means there exists a bijection $\psi : A \rightarrow B$. Know what is meant by “countable” and “uncountable”. Know that the real numbers \mathbb{R} form an uncountable set; know that $\mathbb{Z}, \mathbb{Q}, \mathbb{N}$ are all countable. Know the de Morgan laws of sets, and be able to prove them. Know that the statement “ $A \subseteq B$ ” can be rendered in logical symbols as “ $\forall x(x \in A \implies x \in B)$ ”. Know the de Morgan laws.

- Know what is meant by an equivalence relation; and know what the terms symmetric, reflexive, and transitive mean. Know how to prove that certain relations are equivalence relations – see the note on the course web-page about equivalence relations.

- Know some basic facts about vectors, such as that the length of a vector $v = (v_1, \dots, v_n)$, denoted by $|v|$, is $\sqrt{v_1^2 + \dots + v_n^2}$. Know the definition of parallel and orthogonal vectors.

- Know some basic facts about planes and lines. For example, recall that a plane in n dimensions is the set of all points of the form

$$\{P_0 + tv_1 + uv_2 : u, t \in \mathbb{R}\}.$$

Here, P_0 is a point on the plane and v_1, v_2 are vectors that “sweep out” the plane. Recall that a line can be expressed as the set of all points of the form

$$\{P_0 + tv : t \in \mathbb{R}\},$$

where P_0 is any point on the line.

- Recall that in \mathbb{R}^3 we have an alternate way of expressing the set of points on a plane: A plane is the set of all points $(x, y, z) \in \mathbb{R}^3$ that satisfy

$$ax + by + cz = d,$$

where (a, b, c) is the normal vector of the plane. Recall also how to find a, b, c, d : We can let $(a, b, c) = v_1 \times v_2$, where \times here denotes the cross product.

- Recall some basic facts about cross product. For example, the cross product of vectors (a, b, c) and (d, e, f) can be found by computing the determinant

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix},$$

where i, j, k denote the standard unit vector. Since a, b, c, d, e, f are given constants, this determinant can be written as $t_1i + t_2j + t_3k$, where t_1, t_2, t_3 are scalars.

- Know how to find the distance from a given point to a plane: The nearest point on the plane to the given point will be the projection of the given point onto the plane. There are several ways to find this projection point, but perhaps the simplest is just to find the point P on the plane such that the vector $P - b$ (here, b is the given point) is parallel to the normal vector to the plane. We know that $P - b$ is parallel to the normal vector if and only if the dot product

$$(P - b) \cdot u_i = 0, \quad i = 1, \dots, k,$$

where u_1, \dots, u_k span the plane.

- Know how to compute the volume of a parallelepiped using determinants. One of the key facts that goes in to the derivation of this formula is the fact that

$$|u \times v| = |u||v| \sin \theta,$$

where θ is the angle between the vectors u and v .

- Know the definition of linearly independent vectors. Know the definition of linear dependent. Know how to determine whether a set of vectors is linearly independent (Write down the matrix whose columns are those vectors, and then row reduce. if the rank is n , then they are independent).

- Know how to prove that v_1, \dots, v_n are orthogonal (with respect to any particular inner product) implies that they are linearly independent.

- Know, and know how to prove, some basic facts about dimension. Recall that the dimension of a subspace (or vector space) is the size of any basis for the subspace. It is a central theorem in linear algebra that all bases have the same size. Know how to produce a basis by building up a list of linearly independent vectors until they span all the whole subspace. One of the key ingredients in some of these proofs is as follows: If v_1, \dots, v_n are linearly independent, but do not span all of S , then every vector v outside of the span of v_1, \dots, v_n is linearly independent from v_1, \dots, v_n ; so, any such remaining vector can be adjoined to our list to produce an even larger list v_1, \dots, v_n, v of linearly independent vectors.

- Know the key fact that if S is a subspace of T , and T has dimension n , then S has dimension at most n . Furthermore, S has dimension exactly n if and only if S and T are equal.

- Know that a set of three points in \mathbb{R}^3 are linearly dependent if and only if they lie on the same plane through the origin.

- Know the definition of a vector space (you will not have to know all nine of the axioms, but it is a good idea to know roughly what they are, and how to verify them).

- Know how to prove that V is a subspace of W . Recall that all one needs to show that is: 1) V is non-empty; and, 2) That if $u, v \in V$, then $au + bv \in V$.

- Know some of the standard examples and counterexamples of subspaces. For example: The kernel of a matrix A is a subspace, as is the image of the matrix A (which is $T(V)$, where T is the linear transformation associated to the matrix A ; here, $T : V \rightarrow W$). The set of all polynomials of degree at most n is a subspace, while the set of polynomials of degree exactly n is not a subspace.

- Know the basic properties of inner products. Know how to prove that a function $T : (V, V) \rightarrow \mathbb{R}$ is an inner product on a real vector space. Know the definition of “orthogonality” with respect to this inner product.

- Know a few examples of inner products: For example, the usual dot product is an inner product, as is the integral $\int_0^1 f(x)g(x)dx$, where the underlying vector space is the set of all functions that are integral on $[0, 1]$.

- Know the definition of a norm $\|\cdot\|$: We have that $\|v\| \geq 0$, and equals 0 if and only if $v = 0$; we have the triangle inequality $\|v + w\| \leq \|v\| + \|w\|$.

- Know how to produce a norm given an inner product. In the case where the inner product is defined over the real numbers (as we have done above), then we have that $\|u\| = \langle u, u \rangle^{1/2}$, where $\langle a, b \rangle$ is our inner product between the vectors a and v .

When we have an inner product defined for our vector space, we say that the space is an “inner product space” or “Hilbert Space”; and, when we have a norm for the vector space, then we say that it is a “normed vector space”.

- Know some basic facts, as well as how to prove them, about inner products: We have that a, b are orthogonal with respect to the inner product if and only if $\|a + b\|^2 = \|a\|^2 + \|b\|^2$. Also know the Cauchy-Schwarz inner product, which says that $|\langle a, b \rangle| \leq \|a\| \|b\|$ (here, the norms are those induced by the inner product).

- Know how the Gram-Schmidt process works to construct an orthogonal basis for a subspace.

- Know what it means for T to be a “linear transformation” from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ (it means that for vectors x_1, \dots, x_n and scalars a_1, \dots, a_n , $T(a_1x_1 + \dots + a_nx_n) = a_1T(x_1) + \dots + a_nT(x_n)$). Know how to prove that a given mapping is a linear transformation. Also, know how to solve the following type of problem: Suppose that T is given to be a linear transformation, and the values of $T(x_1), \dots, T(x_n)$ are known; then, find $T(v)$ where v is some linear combination of x_1, \dots, x_n .

- Know that multiplying a matrix times a column vector is a linear transformation.

- Know how to interpret a linear transformation in terms of matrices: Given T , and given a basis v_1, \dots, v_k for V , form the matrix A whose columns are $T(v_1), \dots, T(v_k)$. Then, given a k -dimensional vector $x = (x_1, \dots, x_k)$, written as a column vector, interpret Ax to be $x_1v_1 + \dots + x_kv_k$. This is how matrix multiplication is usually *defined*.

- Know what it means for a column vector to represent a vector with respect to a given basis. For example, if v_1, v_2, v_3 is one basis for a subspace, and we say that “ $x = (x_1, x_2, x_3)$ (written as a column vector) is a vector with respect to the basis v_1, v_2, v_3 ”, we mean that the vector is $x_1v_1 + \dots + x_3v_3$.

- Know how to write the matrix for T with respect to different bases: Perhaps it is initially given with respect to the standard bases, and then you want to switch to different sets of bases for both the starting and ending spaces.

- Know the definition of the orthogonal complement of a subspace S . This orthogonal complement is denoted by S^\perp , and equals the set of all vectors $s' \in \mathbb{R}^n$ (\mathbb{R}^n is the ambient space) that are orthogonal to every vector of S ; that is, $s' \in S^\perp$ if and only if $s' \cdot s = 0$ for every $s \in S$.

- Know the basic fact that $\mathbb{R}^n = S + S^\perp$; that is, for each $v \in \mathbb{R}^n$ there exists $s \in S$ and $s' \in S^\perp$ such that $v = s + s'$.

- Not only does $\mathbb{R}^n = S + S^\perp$, but in fact if the dimension of S is k , then the dimension of S^\perp is $n - k$; so, we have

$$\dim(S) + \dim(S^\perp) = n.$$

Thus, not only can every vector $v \in \mathbb{R}^n$ be written as $s + s'$ as above, but, in fact, the $s \in S$ and $s' \in S^\perp$ must be unique.

- Know what is meant by the projection of a vector v onto a subspace S : Write v uniquely as $s + s'$, $s \in S$ and $s' \in S^\perp$. Then, this s is the “projection of v onto S ”.

Another way to find this projection is as follows: Find $s \in S$ such that $v - s$ is orthogonal to every basis vector of S .

- Know basic facts about matrices, such as: If we say that “ A is an $m \times n$ matrix”, then we mean that it has m rows and n columns. When we write $A_{i,j}$ we mean the entry in the i th row and j th column.

- Know the different ways of describing matrix multiplication: It can be described in terms of dot products (a row vector dot a column vector), or linear combinations of column vectors.

- Know some basic facts about matrix multiplication and addition. Here are a few: If A and B are both $m \times n$ matrix, then the matrix $C = A + B$ is the $m \times n$ matrix whose entries are the sums of respective entries of A and B ; matrix multiplication is associative $A(BC) = (AB)C$; we have a distributive rule $A(B + C) = AB + AC$; finally, if c is a scalar, then cA is the matrix gotten by multiplying every element of A by c .

- Know the connection between the composition of linear transformations and matrix multiplication.

- Know how to solve a system $Ax = b$ using row reduction. Know how to find a “one-to-one parameterization” of the solution set.

Know what is meant by “Row-reduced echelon form”; know how to find

it; and know how to use it to parameterize solutions to $Ax = B$.

- Know the definition of the kernel of a matrix. Know how to find a one-to-one parameterization of the kernel using row reduction.

- Know how to find the inverse of a matrix by putting the left-side of the augmented matrix $[A|I]$ into row-reduced *echelon* form (the right-hand-side will then be the inverse) – note the emphasis on the word echelon.

- Know some basic facts about matrix inverses, such as: If A has both a left-inverse and a right-inverse, then A must be a square matrix and the left-inverse and right-inverse must be equal. Also, $A = n \times n$ is invertible (i.e. has both a left and right inverse) if and only if it has trivial kernel (i.e. the kernel consists only of the 0 vector) if and only if it has rank n .

- Know the definition of the image of a matrix A , denoted $\text{img}(A)$. It is the set of all output vectors $y = Ax$; that is, it is $\{Ax : x \in \mathbb{R}^n\}$. We can interpret this in terms of linear transformations thusly: If we let $T : V \rightarrow W$ be the linear transformation corresponding to A , then $T(V)$ is the image of A .

- Know that if you take a maximal collection of linearly independent column vectors from the matrix A , then you have a basis for the image of A . This maximal number of columns equals the number of pivots when A is row reduced, and it also equals the dimension of the image of A . For obvious reasons, sometimes we refer to the image of A as the “column space of A ”.

- Know that the dimension of the kernel of A is called the nullity of A , and denoted $\text{nullity}(A)$. Know that the dimension of the image of A (and its associated linear transformation T), is called the “rank of A (or T)”.

- Know the rank-nullity formula: Given an $m \times n$ matrix A we have that $\text{rank}(A) + \text{nullity}(A) = n$. We stated this more abstractly in terms of linear transformations T ; but it applies to matrices as well, since matrices can be interpreted as linear transformations.

- Know how to apply the rank-nullity formula to some basic problems.

- Know how to use the rank-nullity formula to prove that

$$\dim(S) + \dim(S^\perp) = n,$$

where \mathbb{R}^n is the ambient space that S and S^\perp lie in. The proof amounts to letting A be the matrix whose rows are a basis for S . Note that if A has size $m \times n$, then we are saying that the ambient space is \mathbb{R}^n and that $\dim(S) = m$. The kernel of this matrix is S^\perp ; and so, $\text{nullity}(A) = \dim(S^\perp)$.

- Know basic properties of determinants: A determinant function is a mapping for the set of $n \times n$ matrices A (with possibly complex number entries) to the complex numbers, and satisfies three axioms. Know the standard axioms (not the ones in the book): d is multilinear, alternating, and assigns the identity matrix value 1.

- Know that d is unique, and therefore we denote it as $\det(A)$, as in “THE determinant of A ”.

- Know how to find the determinant of a matrix by expanding in terms of matrix minors.

- Know how to calculate the determinant by row-reduction. The determinant will equal $(-1)^k$ times the product of diagonal entries in the row reduced version of A , where k is the number of row interchanges needed to row reduce. Here, the only row operations allowed are: 1) Interchange two rows; 2) Add a multiple of one row to another.

- Know what the Hermitian conjugate and transpose of a matrix are; know that they are denoted by A^* and A^t , respectively. Know properties of these; e.g. that $(AB)^t = B^t A^t$ and $(AB)^* = B^* A^*$. Know that for an $n \times n$ square matrix $\det(A) = \det(A^t)$.

- Know the definitions of “eigenvalues”, “eigenvectors”, “diagonalizable”, “unitary”, “Hermitian matrix”, “symmetric matrix”. Know that eigenvectors associated to different eigenvalues are linearly independent; know the proof. Know that if A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable; know how to prove it.

- Know what the characteristic polynomial is, and how to compute it.

- Know how to find eigenvalues and eigenvectors. Know the proof of the fact that if A and B are similar (i.e. $A = C^{-1}BC$), then they have the same characteristic polynomial; and therefore the same eigenvalues. Know that A and A^t have the same eigenvalues.

- Know the statement of the Cayley-Hamilton Theorem. Be able to apply it (e.g. to show that I, A, A^2, \dots, A^n are linearly dependent, given that A is an $n \times n$ matrix.)

- Know the statement of the Spectral Theorem. Be able to apply it – e.g. to show that an $n \times n$ matrix A has the property that $A^t A$ is diagonalizable.