

How to solve some types of Abstract Vector Spaces problems

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In this little note I thought I would explain how to solve certain problems from the course. This might help you to study for the final.

1. Suppose you are asked a question about proving that a logical inference is true or not. If one is allowed to, and if there are only a small number of (true/false) variables involved, I would just write down a truth table. Note that if there are k variables or k propositions, then there should be 2^k rows in the table; the first k columns of the table should contain the true/false values of those k variables/elementary propositions.
2. If you are asked about proving that something is an equivalence relation, don't sweat – it's likely to be EASY; it will be mechanical. e.g. say you have an equivalence relation on $n \times n$ matrices, where $A \sim B$ if there exists an $n \times n$ matrix C such that $A = C^{-1}BC$. Then, clearly $A \sim A$ (reflexive) on taking $C = I$ (identity matrix); if $A \sim B$, then $A = C^{-1}BC$, which means $(C^{-1})^{-1}AC^{-1} = B$, which means $B \sim A$; and, if $A \sim B$ and $B \sim D$, then $A = C^{-1}BC$ and $B = E^{-1}DE$, so that $A = C^{-1}(E^{-1}DE)C = (EC)^{-1}D(EC)$, and therefore $A \sim D$.
3. Induction proofs can be a little tricky; and the source of the trickiness is usually getting things into the right form so that you can apply the induction hypothesis. In the back of your mind should be "how can I relate this to the induction hypothesis".

Another common source of error is that people simply forget how an induction proof is supposed to work. If this is the case with you, then I advise you to try to remind yourself how this simple identity is proved: $1 + 2 + \cdots + n = n(n+1)/2$.

4. The stuff about lines and planes is fairly easy. The part about properties of dot-products and cross-products is a little more tricky. Recall that dot-products can be used to compute projections; determine equations of planes; compute the point on a plane nearest to a given point; and compute the cosine of the angle between vectors. The cross-product is limited to 3D, but it can be ALSO be used to find the equation of plane; compute a vector orthogonal to two given ones; and compute the sine of the angle between vectors.
5. Proving something's a vector space is fairly easy (given the 10 or so axioms); and proving something is a subspace is easier still. It's good to have in mind a couple of examples of things that are NOT vector spaces or subspaces – one example is "all polynomials of degree k ". The problem here is that this set is not closed under linear combinations; and furthermore it doesn't contain the 0 polynomial. If you make it "all polynomials of degree LESS THAN OR EQUAL to k ", THEN you have a vector space.
6. rank-nullity proofs can be TRICKY. If you get stuck on one of these problems, it is probably best to try to construct some examples and then see they help you spot where you are confused. Here are some things to keep in mind: when you compose maps, the rank can only go down; that is, the rank of ST is always less-than-or-equal the rank of T or of S . Also, the kernel can only go up, when comparing T to ST ; that is, the kernel of ST is always at least as big as the kernel of T . But is there a relationship between the kernel of S and the kernel of ST ?
7. How do you show that dimension is a well-defined concept? (i.e. that two bases for a space have the same number of elements.) This is via the following theorem: if S is spanned by the k vectors v_1, \dots, v_k , then any $k + 1$ vectors in S are dependent. From this basic theorem we deduce that if v_1, \dots, v_k are linearly independent vectors contained in $L(\{w_1, \dots, w_\ell\})$, then $k \leq \ell$; likewise, if w_1, \dots, w_ℓ are linearly independent and contained in $L(\{v_1, \dots, v_k\})$, then $\ell \leq k$ – so, if $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_\ell\}$ are two bases, then $k \leq \ell$ and $\ell \leq k$, implying $k = \ell$; and therefore that dimension is well-defined.
8. Suppose I give you a problem about projections where I tell you that

some space S is spanned by the linearly independent vectors v_1, \dots, v_k , and then I ask you to find the projection of x onto S . What should you do first? You should first look to see that these vectors v_1, \dots, v_k are orthogonal or not – if they aren't, then you should first apply Gram-Schmidt to them. Once you've done that, and once you have produced the vectors w_1, \dots, w_k , the projection is then just

$$\frac{\langle x, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \dots + \frac{\langle x, w_k \rangle}{\langle w_k, w_k \rangle} w_k.$$

9. Suppose you know that the determinant of an upper triangular matrix is the product of the diagonal entries. How could you show the same for lower-triangular matrices? You could show it by using the fact that the determinant of a matrix equals the determinant of its transpose.
10. Let V be the vector space of all polynomials of degree at most k , and then let $T : V \rightarrow V$ be the derivative map – i.e. $T(f) = f'$. What are the (complex) eigenvalues of T ? Well, let's see: since T has non-trivial kernel (constants are mapped to 0) we know that 0 is an eigenvalue. Could there be any others? I think not, since if $T(f) = \lambda f$, then we must have $f' = \lambda f$, and that just can't be since f and f' have different degree. And, in fact, if you represent T using a matrix (as we did in class), and then you compute the characteristic polynomial, you will find that it is of the form $(-1)^k x^k$, which only has $x = 0$ as a root.
11. Suppose you know that an $n \times n$ matrix A has n distinct eigenvalues. So, it can be diagonalized as $A = C\Lambda C^{-1}$. Must the C here be uniquely determined by A and Λ ? – that is, could there exist D such that $A = D\Lambda D^{-1}$, with $D \neq C$? Sure – just let the columns of D be non-zero scalar multiples of the columns of C . Is there any other type of D that will work here? No – and the reason is that the columns of C are eigenvectors of A ; and we know that since the eigenvalues of A are distinct, then $Av = \lambda v$ and $Aw = \lambda w$ together imply that v and w are scalar multiples of one another.
12. Perhaps I will have some more later (don't count on it)...