Study Sheet for Math 2406 Midterm 1, Fall 2012

November 7, 2012

Know all the material from the first exam, and then also the following:

1. Know how to prove that something is a vector space (relative to a given set of scalars), given a list of the axioms of a vector space (I won't expect you to be able to memorize the complete list; but will expect you to be able to apply the axioms); know how to prove the uniqueness of the 0-vector; know how to prove the uniqueness of additive inverses $-x$; know how to prove basic properties of a vector space (such as those in Theorem 3.3, page 95). Know some examples of a vector space (such as all function $f : \mathbb{R} \to \mathbb{R}$ vanishing at $x = 1$; all polynomials; all planes through the origin; etc.). Know what is meant by a subspace $S \subseteq V$ contained in a vector space V; and know how to prove that S is a subspace (just show it is non-empty and satisfies the closure axioms).

2. Know what is meant by a set of independent vectors; know what dependent means (in the context of vectors). Know that example about $u_i(x) = e^{a_i x}$ – these are independent functions in the vector space of all functions from $\mathbb{R} \to \mathbb{R}$ if the a_i's are all distinct; know the proof of this fact.

3. Know that any set of independent vectors can be extended to a basis. That is, if $\{x_1, ..., x_k\}$ are independent in a space of dimension n, then there exist $x_{k+1}, ..., x_n$ such that $\{x_1, ..., x_n\}$ is a basis.

4. Know what is meant by an inner product. There are 4 axioms – know them. Also know what is meant by a complex inner product (in particular, know the Hermitian symmetry property). Know what is meant

by a "Euclidean space" (basically, it is a vector space equippend with inner product).

5. Know the standard examples of inner products: the dot product; the integral inner product $(f, g) = \int_a^b f(x)g(x)dx$; know the weighted versions of these; etc.

6. Know the general Cauchy-Schwarz inequality $|(f, g)|^2 \leq (f, f)(g, g)$.

7. Know how to construct a norm $\|f\|$ from an inner product: you can just define $\|f\| = (f, f)^{1/2}$. A "norm" has the defining properties: $\|x\| > 0$ if $x \neq 0$; $\|x\| = 0$ if $x = 0$; $\|cx\| = |c|\|x\|$; and, the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$. Know how to prove the triangle inequality using the Cauchy-Schwarz inequality.

8. Know what is meant by orthogonal vectors. There are basically two equivalent definitions you could use: definition 1 is that $x_1, ..., x_k$ are orthogonal if all pairwise inner-products (x_i, x_j), $i \neq j$, are 0; and definition 2 is that those vectors satisfy a kind of Pythagorean Theorem, in that $\|x_1 + \cdots + x_k\|^2 = \|x_1\|^2 + \cdots + \|x_k\|^2$.

9. Know some examples of orthogonal vectors – like, for instance, the functions $e^{2\pi it}$, with respect to the inner product $(f, g) = \int_0^1 f(x)\overline{g(x)}dx$.

10. Know that if $x_1, ..., x_n$ are orthogonal, then they must be linearly independent. Obviously, however, not every set of linearly independent vectors is orthogonal.

11. Know the definition of orthonormal. Know that if $v_1, ..., v_n$ are orthonormal, then any vector x in the span of those vectors may be written as $x = \sum_i (x, v_i)v_i$. Know Parseval's formula: $(x, y) = \sum_i (x, v_i)\overline{(y, v_i)}$. Know the proofs of these facts.

12. Know how the Gram-Schmidt process works to produce an orthogonal set of vectors spanning the same space as a given set of vectors. Be able to do examples of applying Gram-Schmidt.

13. Know the definition of the projection of a vector x onto a subspace S. Know how to compute it, given S and the inner-product. Letting $p(x)$ denote this projection, know that $p(x)$ has the property that $\|p(x) - x\| \leq \|y - x\|$, for any $y \in S$; that is, $p(x)$ comes closer to x than any

other vector in S. Know what this means in terms of Fourier series (see the book); know how to find the gram-schmidt basis for a given space of polynomials, given a particular inner-product.

14. Know what a linear transformation is, and know how to prove or disprove that various maps $T : V \to W$ are (or are not) linear transformations.

15. Know the definition of nullspace. Know the fact that if $T : V \to W$, and V has dimension n, then

$$\text{rank}(T) + \text{nullity}(T) \;=\; n.$$

It might also be a good idea to have some basic understanding of how this is proved.

16. Know what the "orthogonal complement" $S^{\perp}$ means. Know how to apply rank-nullity to show that $\dim(S) + \dim(S^{\perp}) = n$, where $S \subseteq V$, $\dim(V) = n$.

17. If f is a linear transformation, know the fact that f injective if and only if the kernel of f is trivial ("kernal" means the same thing as nullspace $N(f)$) if and only if the nullity of f is 0. What does knowing that f surjective tell you about the rank of f? What does knowing that f bijective tell you about the relationship between the dimension of V and the dimension of W?

18. Know the definition of $L(V, W)$ – it is the set of linear transformations from $V \to W$. Know that this set of linear trasformations is itself a vector space, where $+$ is defined as follows: if f, g are linear transformations, then the sum $h = f + g$ is the map that sends

$$x \;\to\; h(x) = (f + g)(x) = f(x) + g(x).$$

This map h is also a linear transformation. Scalar multiplication is defined in the obvious way. There is one additional operation that you can do on linear transformations, and that is compose them: if $f : V \to W$ and $g : W \to X$, are linear transformations, then the composition $h = gf = g \circ f$, which maps

$$x \;\to\; h(x) = (gf)(x) = g(f(x))$$

is a linear transformation from $V \to X$.

19. Know that dim$L(V, W) = nm$ if $n =$ dim$V$ and $m =$ dim$W$.

20. Know how to represent a linear transformation $T : V \to W$ as a matrix. Know that if you change the basis for V and the basis for W, this matrix also changes; know how to find the matrix for T w.r.t. different bases. Know how to use change-of-basis matrices.

21. Know the fact that if $T : V \to W$ is a surjective linear transformation then the set of all $x \in V$ such that $T(x) = y$ has the form $x = x_0 + t$, where $t \in N(T)$. Know how to use this to solve systems of linear equations. Know how to put a matrix into Row Reduced Echelon Form, and then know how to use that form to find a basis for $N(T)$ and to find x_0 (given y, of course).