Math 3012, Midterm 2, Fall 2009 Study Guide

November 11, 2009

- Know the well-ordering principle, and the meaning of "Type I" and "Type II" induction. Know how to give some basic induction proofs – e.g. look over the examples in section 4.1.
- Know the basic properties of divisibility, such as: if $a|b, b|a$, and $a, b \geq 1$, then $a = b$; and, if $a|b$ and $a|c$, then $a|(\lambda b + \mu c)$ for arbitrary integers λ and μ. Know how to find the $\gcd(a, b)$ for integers a, b, and know how to use Knuth's algorithm for find integers x and y such that $ax + by = \gcd(a, b)$. Know the fundamental theorem of arithmetic, and more-or-less how it is proved.
- Know basic properties of congruences (e.g. if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$). Know how to solve for x in a congruence equation such as $ax \equiv b \pmod{n}$.
- Know the definiton of a relation $A \to B$ in terms of cross products $A \times B$. Know the definition of a function $f : A \to B$ in terms of a subset $f \subseteq A \times B$ having certain properties. Know the definition of injective (one-to-one), surjective (onto), and bijective (both one-to-one and onto) functions. Know basic properties, such as that the composition of injective functions is injective, etc. Know that f is invertible if and only if f is a bijection. Know that if A and B are finite sets, and that $f : A \to B$ is an injection, then $|A| \leq |B|$; likewise, know if that $f : A \to B$ is a surjection, then $|A| \geq |B|$. Know how to count the number of injective functions $f : A \to B$.
- Know how to count the number of surjective functions $f : A \to B$ by relating it to Stirling numbers of the second kind. Know how to compute Stirling numbers using the functional relation for $S(m, n)$ (ways

of partitioning m objects into n parts, which is what Striling numbers count) given by

$$S(m+1,n) \;=\; S(m,n-1) + nS(m,n).$$

Know how to derive this formula.

- Know the meanings and uses of some special functions, like the projection

$$\begin{aligned} \pi_A \;:\; A \times B &\;\rightarrow\; A \\ (a,b) &\;\rightarrow\; a. \end{aligned}$$

and the identity mapping

$$\begin{aligned} 1_A \;:\; A &\;\rightarrow\; A \\ a &\;\rightarrow\; a \end{aligned}$$

- Know the meaning of reflexive, symmetric, transitive, and antisymmetric. Know the definition of an "equivalence relation" – it is a relation that is reflexive, symmetric and transitive. Be able to prove that certain relations are in fact equivalence relations. Know the definition of a partial ordering on a set (a set with such an ordering is called a poset) – reflexive, antisymmetric and transitive. Know some examples of posets, and how to prove that they are in fact posets. Know how to construct a total ordering that is consistent with the ordering in the poset, in the sence that if $\lesssim$ indicates the ordering in the poset, while $\leq$ indicates the ordering in the total ordering, then

$$a \lesssim b \implies a \leq b.$$

(But vice versa need not hold.) Know how to construct the Hasse diagram for a poset, and know how to run the "topological sorting algorithm" to construct the total order.

- Know the pigeonhole principle, and be able to prove some basic results using it. See all the examples in section 5.5 for some good examples.

- Know what a finite state machine is (my definition, not the book's): You have an alphabet Σ, a single start state (usually denoted q_0), possibly several halt states, and possibly some other states, along with

some "transition rules". Know how to build a finite state machine that accepts a given "regular language". Know how to write down the adjacency matrix for the machine; and know how to express the number of strings of a given length in the langauge, as a function of entries in the nth power of this adjacency matrix.

- Know what a linear recurrence sequence is, and know how to express the nth term in the sequence in terms of powers of certain matrices. Know how to write down the generating function for the linear recurrence sequence.

  NOTE: I will *not* hold you responsible for actually solving for the formula of a linear recurrence in terms of the roots of characteristic polynomials, as that would be too difficult for a 1 hour test. *However, I may ask you some questions that test whether you know how to do it.*