Applied Combinatorics Final Exam

September 17, 2009

1. Use mathematical induction to prove that every integer $n \geq 1$ can be written as a sum of distinct powers of 2 of the form 2^a, where $a \geq 0$ is an integer. For example, $1 = 2^0$, $2 = 2^1$, $3 = 2^0 + 2^1$, $4 = 2^2$, $5 = 2^0 + 2^2$, $6 = 2^1 + 2^2$, $7 = 2^0 + 2^1 + 2^2$, and so on.

2. Show that if one picks 10 points in a 3×3 square, two of these points must be at most $\sqrt{2}$ apart.

3. Determine the number of strings of length 4 one can make using the letters from the string ABRACADABRA. Alternatively, such strings are all those of length 4 using only the letters A, B, C, D, and R, containing at most 4 A's, 2 B's, 1 C, 1 D, and 2 R's.

4.

a. Determine the rook polynomial for a certain chess board described as follows: The board is a subset of the squares in a 4×4 square consisting of the square

$$(1,1), (1,2), (2,1), (2,2), (2,3), (3,3), (3,4), (4,3), (4,4).$$

(My notation here (a, b) refers to the square in row a, column b.) Recall the recurrence formula for a rook polynomial: $R(x) = xR_1(x) + R_2(x)$, where $R_2(x)$ is the rook polynomial for the board after deleting a square, and where $R_1(x)$ is the rook polynomial after deleting that same square as well as all the squares in the same row and column.

b. Determine the number of injective functions from $A = \{1, 2, 3, 4\}$ to itself satisfying the following restrictions: $f(1)$ cannot be 3 or 4; $f(2)$ cannot be 4; and, $f(3)$ and $f(4)$ cannot be 1 or 2.

5. Suppose A and B have 5 elements.

a. Define surjective, injective, bijective.

b. How many injective functions are there from A to B?

c. How many surjective functions are there from A to B? (this is easier than you think – don't use Stirling numbers.)

6. Determine the number of integers $x_1, x_2, x_3, x_4 \geq 0$ satisfying

$$20 \;=\; x_1 + 2x_2 + 3x_3 + 3x_4.$$

7. Consider the recurrence relation

$$X_{n+1} \;=\; 3X_n - X_{n-1} - X_{n-2}.$$

It is known that for $X_0 = X_1 = 0$ and $X_2 = 1$ that the sequence $X_0, X_1, ...$ has the property that

$$\lim_{n \to \infty} \frac{X_{n+1}}{X_n} \;=\; \theta,$$

for some constant $\theta > 0$. Determine θ.

8.

a. Determine the number of permutations of the string

$$AABBCCCDDEEEE.$$

b. Determine the number of strings of 9 digits (a digit is a number 0 through 9) having three 0's, none of which are consecutive. (So, for example, you cannot have 134005609.)

9. Determine the number of ways of partitioning the set $A := \{1, 2, 3, 4, 5, 6\}$ into 3 non-empty subsets. Recall the recurrence for Stirling numbers: $S(m, n) = nS(m-1, n) + S(m-1, n-1)$.

10. Determine the number of ways of coloring the vertices of a certain graph G (defined below) using r colors, so that no two adjacent vertices (vertices connected by an edge) share the same color. The graph G is defined to be the one where we take K_5 and delete the edge from vertex 1 to vertex 2. Recall that the chromatic polynomial $P_G(\lambda)$ satisfies the recurrence

$$P_G(\lambda) \;=\; P_K(\lambda) - P_H(\lambda),$$

where H is the graph gotten by taking G and contracting along some edge e, and K is the graph gotten by taking G and removing the edge e. (Hint: Besides using the recurrence relation, another way to solve the problem is to use the fact that $P_G(0) = P_G(1) = P_G(2) = P_G(3) = 0$, since G cannot be 3-colored. If you also had $P_G(4)$ [which is non-zero], then you could easily determine the chromatic polynomial.)