A few solutions to some homeworks

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1 Homework 4, part 2

2.

a. We know that

$$\gcd(\Delta x + 1,\ \Delta) \ = \ \gcd((\Delta x + 1) \bmod \Delta,\ \Delta) \ = \ \gcd(1,\ \Delta) \ = \ 1.$$

b. Likewise,

$$\gcd(\Delta + x,\ \Delta) \ = \ \gcd((\Delta + x) \bmod \Delta,\ \Delta) \ = \ \gcd(x, \Delta).$$

Then, since x is a prime, this last gcd is either 1 or x. But since $x > 7$ is prime, it has no common factors with $2, 3, 5$, or 7; so, this last gcd is 1.

3. Let x denote the number of type A coins that Alice has, and let y denote the number of type B coins that Bob has. Then, the value of Alice's coins is $55x$, and the value of Bob's coins is $101y$. We seek x and y so that

$$101y - 55x \ = \ 1.$$

We will instead just solve

$$101a + 55b \ = \ 1.$$

To do this, we use Knuth's algorithm:

$$\begin{bmatrix} 1 & 0 & 101 \\ 0 & 1 & 55 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 46 \\ 0 & 1 & 55 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 46 \\ -1 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 6 & -11 & 1 \\ -1 & 2 & 9 \end{bmatrix}.$$

So, from the first line of the last box we conclude that

$$101 \cdot 6 - 55 \cdot 11 \; = \; 1.$$

This means that

$$y \; = \; 6, \; x \; = \; 11,$$

and this solution turns out to be unique, given the range of x and y supplied by the problem.

2 Homework 4, part 3

2. We note that if $z \equiv 1, 2, 4 \pmod 7$, then $z^3 \equiv 1 \pmod 7$; if $z \equiv 3, 5, 6 \pmod 7$, then $z^3 \equiv -1 \pmod 7$; and, lastly, if $z \equiv 0 \pmod 7$, then $z^3 \equiv 0 \pmod 7$. What this means is that

$$x^3 + y^3 \; \equiv \; 0, 1, -1, 2, \text{ or } -2 \pmod 7;$$

and therefore

$$x^3 + y^3 + 3 \; \not\equiv \; 0 \pmod 7.$$

It follows that 7 can *never* divide $x^3 + y^3 + 3$ for any integers x, y.