Midterm 2, Math 3012, Fall 2009

November 12, 2009

NOTE: You have 1 hour to complete this exam.

1. (Trivial) Define the following terms.
a. Antisymmetric relation.
b. State the fundamental theorem of arithmetic.
c. State the Well-Ordering Principle.
d. bijection.
e. Explain what Stirling numbers of the second kind count.

2. (Easy) Let $U = \{1, 2, 3, 4\}$, and let R be a relation on the powerset 2^U of the set U, such that if $a, b \in 2^U$, then xRy if and only if $x \subseteq y$. This turns out to be a partial order.
a. Construct the Hasse diagram for this partial order.
b. Use the topological sorting algorithm to create a total order '$\leq$' on the set 2^U, consistent with R, such that

$$xRy \implies x \leq y.$$

3.(Easy/Medium) Assume that dimes weigh 23 decigrams and nickels weigh 45 decigrams. A collection of dimes and nickels weighs 1568 decigrams in total. Determine the number of dimes and the number of nickels in the collection.

4.(Medium/Difficult) Let $A = \{a_1, a_2, ...\}$ be an infinite set of real numbers satisfying

$$1 \leq a_1 < a_2 < a_3 < \cdots, \text{ with } a_{i+1}/a_i \geq 2.$$

Use mathematical induction to show that for each integer $k \geq 1$, all the k-fold sums

$$b_1 + b_2 + \cdots + b_k,\ \ b_1 < \cdots < b_k,\ \text{all } b_i \in A,$$

are distinct. (Hint: Induct on k. Show that if two sums are equal, their largest terms...)

5.(Difficult) Suppose that $x_1, x_2, ..., x_n$ are $n \geq 2$ real numbers that belong to the interval $[0, 1]$. Prove that there is a linear combination

$$\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n \ \in\ [0, n/2^n],$$

where all the $\varepsilon_i = 0, 1$ or -1. (Hint: Pigeonhole principle. Think about what happens when you subtract two sums

$$(x_{i_1} + x_{i_2} + \cdots + x_{i_k}) - (x_{j_1} + x_{j_2} + \cdots + x_{j_\ell}),$$

and then think about expressing it as a linear combination...)