Solutions to Math 3012, Midterm 2, Fall 2009

November 12, 2009

1. You can look these up yourself.

2. This is a straightforward exercise, so I will not bother to say how to do it. Also, answers vary on part b (there are multiple ways of running the topological sorting algorithm).

3. The thing that is interesting about this problem is that it shows how a single number can be used to determine both the number of dimes and number of nickels, which seems a little counter-intuitive.

The idea for how to solve it is to let x be the number of dimes and y be the number of nickels, and notice that

$$23x + 45y \ = \ 1568.$$

Now mod both sides out by 45, and notice that the equation becomes

$$23x \ \equiv \ 38 \pmod{45}.$$

Now, since $2 \cdot 23 \equiv 1 \pmod{45}$ (this number 2 can be found by using Knuth's algorithm), we have that

$$x \ \equiv \ 2 \cdot 23x \ \equiv \ 2 \cdot 38 \ = \ 76 \ \equiv \ 31 \pmod{45}.$$

So, $x = 31, 76, 121, ...$, and of course 76 is already too big, because we would have

$$23 \cdot 76 \ = \ 1748 \ > \ 1568.$$

So, $x = 31$, and then you solve for y to get $y = 19$.

3. The claim clearly holds for $k = 1$, for all the a_i are distinct. Assume, for proof by induction, that the claim holds for $k = n$. Now we prove it for $k = n+1$: Suppose that

$$b_1 + \cdots + b_{n+1} \ = \ b'_1 + \cdots + b'_{n+1}, \tag{1}$$

where the b_i's and b'_i's are drawn from the set of the a_i's, and

$$b_1 < b_2 < \cdots < b_{n+1}, \text{ and } b'_1 < b'_1 < \cdots < b'_{n+1}.$$

Assume, without loss of generality, that $b'_{n+1} \geq b_{n+1}$ (either this holds, or $b_{n+1} \geq b'_{n+1}$, in which case we could just slightly rewrite the argument below). We will show, in fact, that $b'_{n+1} = b_{n+1}$, meaning that we can remove both these terms from (1), leaving us with an equation of two sums of n terms each. By the induction hypothesis we will then be able to conclude that $b_i = b'_i$, $i = 1, ..., n+1$, which will prove the induction step.

So, suppose, on the contrary, that $b'_{n+1} \neq b_{n+1}$. Then, we must have that $b'_{n+1} > b_{n+1}$ (remember... we are assuming $b'_{n+1} \geq b_{n+1}$); therefore, we may assume that $b'_{n+1} = a_r$, for some r, while $b_{n+1} \leq a_{r-1}$. It follows that

$$b_1+\cdots+b_{n+1} \ \leq \ a_1+\cdots+a_{r-1} \ \leq \ a_{r-1}(1+1/2+1/4+\cdots+1/2^{r-2}) \ < \ 2a_{r-1} \ \leq \ a_r = b'_{n+1},$$

which contradicts (1), and so we are done.

Note that here we have used the fact that

$$a_{r-2} \leq a_{r-1}/2, \ a_{r-3} \leq a_{r-2}/2 \leq a_{r-1}/4, \ ..., \ a_1 \leq a_{r-1}/2^{r-2}.$$

You can think of the proof I just gave as a slight generalization of the one I worked out in class showing that binary number expansions are unique.

5. First, if all the $x_i \geq 1 - n/2^n$, then we are done since the difference between a pair of them will be in $[0, n/2^n]$. So, suppose that at least one of the $x_i < 1-n/2^n$. Then, the sum of all the x_i must be smaller than $n-n/2^n$. Now, take this interval $[0, n - n/2^n]$, and break it up into $2^n - 1$ equal-sized subintervals of width $(n - n/2^n)/(2^n - 1) = n/2^n$. Now we do a pigeonhole principle argument:

To each of the 2^n subsets S of $\{x_1, ..., x_n\}$, let $\Sigma(S)$ denote the sum of the elements in S (we adopt the convention $\Sigma(\emptyset) = 0$). Regardless of which subset S we choose, we must have that $\Sigma(S) \in [0, n - n/2^n]$. Now, since there are 2^n subsets S to choose from, which exceeds the number of our subintervals, by the pigeonhole principle there must exist subsets S and T such that

$$\Sigma(S) \ \le \ \Sigma(T),$$

yet where $\Sigma(S)$ and $\Sigma(T)$ lie in the same subinterval. This means that

$$0 \ \le \ \Sigma(T) - \Sigma(S) \ \le \ (n - n/2^n)/(2^n - 1) \ = \ n/2^n.$$

And now it is clear that $\Sigma(T) - \Sigma(S)$ can be expressed as a linear combination

$$\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n, \ \varepsilon_i \in \{0, 1, -1\}.$$

(For example, if $n = 10$ and $T = \{x_1, x_3, x_5\}$, while $S = \{x_2, x_3, x_6\}$, then $\varepsilon_1 = \varepsilon_5 = 1$, $\varepsilon_2 = \varepsilon_6 = -1$, while $\varepsilon_3 = \varepsilon_4 = \varepsilon_7 = \cdots = \varepsilon_{10} = 0$.)