Study guide for Math 3012 final exam, Fall 2009

December 7, 2009

1. Study basics of counting, including the rule of the sum and rule of the product.

2. Know how to count the number of permuations of r objects chosen from among n things (this is $P(n, r) = n(n - 1)...(n - r + 1)$.) Know the meaning of the number of combinations of n objects chosen from r (This is the number of subsets of size r chosen from among n things. This number is $\binom{n}{r}$.)

3. Know how to count permutations and combinations with repetitions. Know the method of barriers, and how to count the numer of ways of writing $n = x_1 + ... + x_k$. Know the same thing, except where there are some restrictions on the x_i's.

4. Know the definition of the Catalan number

$$C_k = \frac{1}{k+1}\binom{2k}{k},$$

which counts the number strings of $2k$ balanced ()'s. It also counts the numer of lattice paths from lying on or below the line $y = x$.

5. Know the definition of a set, subsets, and elements. Know the meaning of the cross product of sets, the power set of a set, the union of two sets, the difference of two sets (this is $A - B$, which is the elements of A that do not blong to B). Know how many elements are in each type of set. For example, if A has m elements, B has n elements, then AxB has mn elements. The powerset 2^A will have 2^m elements.

6. Know the basics of logic, and how to prove logical statements using truth tables. Know the difference between $A \longrightarrow B$ and $A \implies B$. Know some of the basic rules of manipulating logical expressions – mostly, they are quite similar to statements about sets. Know the de Morgan laws for logic and for sets.

7. Be able to prove a theorem via mathematical induction. Know the division algorithm and the Euclidean algorithm. Be able to find intgers x and y such that $ax + by = g$, where g is the gcd of a and b. Be able to prove basic facts about divisibility, such as if $a|b$ and $b|c$, then $a|c$. Or, maybe something like, if $a^3|b^2$, then $a|b$.

8. Know the means of surjection, injection, bijection. Be able to prove that certain functions are any or all of the three.

9. Know the definition of the Stirling number $S(m, n)$, which is the number of ways of partitioning a set of m objects into n subsets, where no subset is empty. Know the rationship between $S(m, n)$ and the number of surjections from a set of m elements to a set of n elements. Know how to derive the basic recurrence relation for $S(m, n)$ (Namely, $S(m+1, n) = S(m, n-1) + nS(m, n)$.) Know how to count the number of injections and bijections from one set to another.

10. Know how to give an easy pigeonhole principle proof of some result (such as that if you have 51 distinct numbers chosen from among $1, ..., 100$, two of them must be relatively prime).

11. Know the definitions: Binary Relation, Equivalence Relation, Partial Ordering, Commutative, Associative, reflexive, symmetric, transitive, antisymmetric.

12. Know how to construct a Hasse diagram for a partial order, and know how to construct a total order which is compatible with that partial order.

13. Know inclusion-exclusion and how to apply it to some basic counting problems.

14. Know the basics of probability theory, such as the three axioms of probability. Know inclusion-exclusion for probability. Know Bayes's theorem for 2 or more events, and how to apply it to solve problems.

15. Know how to find the generating function for certain basic problems, such as the number of ways of writing an integer n as a sum $n = x_1 + 2x_2 + 3x_3 + 5x_4$, where x_1, x_2, x_3, x_4 are all at least 0.

16. Know how to use generating functions to find the sum of the first several terms in a sequence $a_0 + ... + a_n$, where the sequence is $a_0, a_1, ...$ The generating function for the sequence $b_n = a_0 + ... + a_n$ is $1/(1-x)$ times the generating function for a_n.

17. Know the definition of an exponential generating function, and know how to use it to count the numer of strings of length 4, say, from the string AARDVARK. Any such string can have at most 3 A's, 2 R's, 1 D, 1 V, and 1 K.

18. Know how to find the generating function for a linear recurrence sequence, say, $x_{n+1} = 2x_n - x_{n-1}$ with initial conditions $x_0 = 1$ and $x_1 = 2$. If x_n/x_{n-1} tends to a limit as n tends to infinity, know how to find it. Know the general formula for the nth term of a linear recurrence sequence. This formula is:

$$x_n = p_1(n)y_1^n + ... + p_k(n)y_k^n,$$

where the y_i are roots of the characteristic polynomial, and where the degree of the polynomial $p_i(x)$ is at most $a_i - 1$, where a_i is the multiplicity of the root y_i in the characteristic polynomial.

19. Know the definition of a finite state machine: It is a collection of states, an alphabet, a single start state, and (possibly) multiple halt states, and can be represented by a directed graph. Know how to find the adjacency matrix associated to the machine, and how to use it to compute the number of strings of a given length in the regular language accepted by the machine.

20. Know basic definitions associated to graphs. Know what a planar graph is, know Euler's formula $V - E + F = 2$. Know how to prove some basic things, such as that E is at most $3V - 6$. Know Kuratowski's theorem.

21. Know what an Euler circuit is, and what a Hamiltonian cycle is. Know how to find an Euler circuit.