A few solutions to exam 1, Math 3770, Fall 2008

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Here I will give solutions to problems 2,3, and 4 ONLY – problems 1 and 5 are basic and you can figure them out on your own.

2. There are many ways to solve this one. Here is one of the simplest: There are basically two kinds of marbles – green and not-green. There are 6 green and 18 not-green. The probability you draw at least three marbles is the probability that your first two draws are not-green. So,

$$\begin{aligned}
&P((\text{first not} - \text{green}) \cap (\text{second not} - \text{green})) \\
&\quad = P((\text{first not} - \text{green}))P((\text{second not} - \text{green})|(\text{first not} - \text{green})) \\
&\quad = \frac{18}{24} \cdot \frac{17}{23} \\
&\quad = \frac{51}{92}.
\end{aligned}$$

Another way to solve it is to sum up the possibilities for the first two draws: RB, BR, RR, BB.

3. This problem is easier than it looks. By combining "inclusion-exclusion", together with independence (i.e. $P(A \cap B) = P(A)P(B)$), and some simple algebra, you get:

$$\begin{aligned}
P(A \cup B) = P(A) + P(B) - P(A \cap B) &= P(A) + P(B) - P(A)P(B) \\
&= 1 - (1 - P(A))(1 - P(B)).
\end{aligned}$$

Here is another approach: Since

$$A \cup B = \overline{\overline{A} \cap \overline{B}},$$

we find

$$P(A\cup B) \;=\; 1-P(\overline{A}\cap\overline{B}) \;=\; 1-P(\overline{A})P(\overline{B}) \;=\; 1-(1-P(A))(1-P(B)).$$

Well, there is a subtlety to this approach: Just because we know $P(A\cap B) = P(A)P(B)$, how do we deduce $P(\overline{A}\cap\overline{B}) = P(\overline{A})P(\overline{B})$? This is easy to prove... I will leave it to you to discover how to show it.

4. The given data in the problem is equivalent to the following:

$$\begin{aligned}
P(\text{Black} \mid \text{Local}) &= 0.3\\
P(\text{White} \mid \text{Local}) &= 0.2\\
P(\text{Green} \mid \text{Local}) &= 0.5 \text{ (follows from the two equalities above)}\\
P(\text{Green} \mid \text{College}) &= 0.1\\
P(\text{Local}) &= 0.3\\
P(\text{College}) &= 0.7 \text{ (deduction from previous equality).}
\end{aligned}$$

Bayes's theorem then gives

$$\begin{aligned}
P(\text{Local} \mid \text{Green}) &= \frac{P(\text{Green} \mid \text{Local})P(\text{Local})}{P(\text{Green} \mid \text{Local})P(\text{Local}) + P(\text{Green} \mid \text{College})P(\text{College})}\\
&= \frac{0.5\cdot 0.3}{0.5\cdot 0.3 + 0.1\cdot 0.7}\\
&= \frac{15}{22}.
\end{aligned}$$