Study Sheet for Math 3215 Final Exam, Summer 2009

July 25, 2009

Below is a rough guide to some topics to study for the final exam. Whenever I make such a guide, inevitably I will leave some things out (accidentally), hence the word 'rough'.

- Know the definition of a sample space and probability measure, and know how to set these up, given the description of an experiment such as "you roll two six-sided die" (here the sample space would consist of all pairs (a, b), $a = 1, ..., 6$, $b = 1, ..., 6$). Know how to think of 'events' (to be measured by a probability measure) as corresponding to subsets of S. Know the three basic axioms of probability. Know the basic set-manipulation operations, such as complementation, intersection, union, cross product. Know the definition of $\overline{A}$ – it is the complement of A with respect to some universal set (usually, the universal set is the sample space). Know de Morgan's laws $\mathbb{P}(\overline{A \cup B}) = \mathbb{P}(\overline{A} \cap \overline{B})$, and $\mathbb{P}(\overline{A \cap B}) = \mathbb{P}(\overline{A} \cup \overline{B})$.

- Know the definition of conditional probability $\mathbb{P}(A|B)$. Know the "Chain Rule" of probability, which in the case of 3 events is

$$\mathbb{P}(A, B, C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A, B),$$

  where here $P(U, V, W)$ means $P(U \cap V \cap W)$. Know how to apply the Chain Rule to, for example, the "Birthday Paradox". Know Bayes's Theorem (There are actually different forms of Bayes's Theorem) – and know the compound version that involves more than just 2 events. Know how to apply Bayes's Theorem to some basic problems.

- Know some basic facts about random variables (i.e. know their mean and variance, and how to compute it quickly), such as: the difference between continuous and discrete; the meaning of the probability density function and mass function; the meaning of the cumulative distribution function. Also know the meaning of 'percentile' and how to compute it given the pdf of an r.v.

- Know how to compute the pdf of an r.v. Y that is a function of some other r.v. X – so, $Y = f(X)$, for some function f. Basically, you first find the cdf of Y, and then you find the pdf by taking a derivative.

- Know the definition of expectation, and how to compute it given the pdf (or mass function). Know the "linearity of expectation": if $X_1, ..., X_k$ are r.v.'s, *not necessarily independent*, then

$$\mathbb{E}(a_1 X_1 + \cdots + a_k X_k) \ = \ a_1 \mathbb{E}(X_1) + \cdots + a_k \mathbb{E}(X_k).$$

- Know the following basic types of discrete random variables: Bernoulli, Binomial, Poisson, Uniform (discrete), Geometric. And be familiar with the Negative Binomial, Hypergeometric, and Multinomial.

- Know the following basic types of continuous random variables: Exponential, Gaussian, and Uniform. And be familiar with the follow (be able to do table lookups, and to manipulate pdf or compute moment generating functions): Chi-squared, Student-t, Gamma. Know how to prove that $\int_{-\infty}^{\infty} e^{-x/2} dx = \sqrt{2\pi}$.

- Know basic facts about the gamma function, such as its definition $\int_0^\infty e^{-t} t^{x-1} dt = \Gamma(x)$, that for an integer $n \geq 1$ we have $\Gamma(n) = (n-1)!$, and that $\Gamma(1/2) = \sqrt{\pi}$.

- Know the definition of the moment generating function, and how to compute it for some standard random variables (such as, though exclusively, the following: the Uniform, Normal, Poisson, Chi-squared). Know the different ways that the moment generating function may be used: first, it can be used to compute moments, for if X is a random variable with mgf $M_X(t)$, then the kth derivative of $M_X(t)$ evaluated at $t = 0$ gives the kth moment of X; that is,

$$\mathbb{E}(X^k) \ = \ \lim_{t \to 0} M_X^{(k)}(t).$$

Another way that the kth moment of X may be found is by realizing that, as a power series,

$$M_X(t) \;=\; \sum_{k=0}^{\infty} \frac{\mathbb{E}(X^k)}{k!} t^k.$$

Yet another way that mgf's can be used is to prove that two random variables have the same distribution: they have the same distribution if their mgf's are equal in some neighborhood that includes 0. One last property of mgf's that is important to know is that if X is the sum of *independent* r.v.'s $X_1, ..., X_n$ – so, $X = X_1 + \cdots + X_n$ – then we have that

$$M_X(t) \;=\; M_{X_1}(t) \cdots M_{X_n}(t).$$

- Know the definition of independent random variables. Know how to prove that if $X_1, ..., X_k$ are independent, then $V(X_1 + \cdots + X_k) = V(X_1) + \cdots + V(X_k)$. Know the "translation invariance" property of the variance $V(X + c) = V(X)$, where X is an r.v. and where c is a constant.

- Know the definition of covariance, and know how to prove that $\mathrm{cov}(X, Y) = \mathbb{E}(XY) - \mu_x \mu_y$. Know the fact that if X and Y are independent, then $\mathrm{cov}(X, Y) = 0$; and, know how to produce a counterexample for the converse – i.e. know how to produce X and Y with $\mathrm{cov}(X, Y) = 0$ and yet X and Y are dependent. Know the definition of the "correlation coefficient", and a few of its basic properties, such as that $\rho(X, Y) \in [-1, 1]$, and that if X and Y are "linearly related" – i.e. $Y = \lambda X + \beta$, $\lambda \neq 0$ – then $\rho = \pm 1$ (the sign is determined by the sign of λ). (and, there is almost a converse to this; that is, it is *almost* true that if $\rho = \pm 1$ then X and Y are linearly related).

- Know the follow: joint pdf, marginal pdf, conditional pdf, and condition expectation. Know how to compute the latter three of these given the joint pdf.

- Know the statement of the Law of Large Numbers and the Central Limit Theorem. Know how to apply them to some basic problems, such as the "Noise Cancellation" problem and the "Hypothesis Testing" problem. Also know how to use the CLT to approximate a binomial r.v.

- Know the "chi-squared test statistic" (see the relevant note on the course webpage about it), and how to use it to test a hypothesis about the makeup of a population (given χ^2 tables). Know how to use the CLT approximation of a binomial r.v. to test a population makeup of two types of individuals (such as 'smoker' and 'non-smoker').

- Know the meaning of the terms: null hypothesis (H_0), alternate hypothesis (H_a), test statistic, sample mean, sample variance, Type I error, Type II error, biased and unbiased estimators. Be able to compute the probability of making a "Type I" error using a statistical test.

- Know how to apply the CLT to find confidence intervals for the expected value μ of a r.v. X – basically, this is the "pedagogical example" from class. Know how to find a C.I. for μ and σ^2 when X is normal, using the Student-t and χ^2 distributions (be able to use Student-t and χ^2 tables).