# Homework 1, Math 3225

September 10, 2010

1. Recall that events $A_1, ..., A_k$ are independent if for every non-empty subset $S$ of $\{1, ..., k\}$ we have

$$\mathbb{P}(\cap_{s\in S} A_s) \ = \ \prod_{s\in S} \mathbb{P}(A_s). \tag{1}$$

As a consequence of this, it turns out that this implies, and is equivalent to, the statement

**Claim.** For $i = 1, 2..., k$ we have that for any set $B$ in the $\sigma$-algebra generated by all the sets $A_j$, $j \neq i$,

$$\mathbb{P}(A_i, B) \ = \ \mathbb{P}(A_i)\mathbb{P}(B).$$

In the special case $i = 1$ this would be saying that for any $B \in \sigma(A_2, ..., A_k)$ – i.e. $B$ is any set gotten by doing any number of intersections, unions and complements (which will turn out to be finite in number, since $k < \infty$) of the events $A_2, ..., A_k$ – we must have that $\mathbb{P}(A_1, B) \ = \ \mathbb{P}(A_1)\mathbb{P}(B)$.

This equivalence is somewhat tricky to prove; and to give you a taste of what is involved, your first problem will be to prove the following: Suppose that $A, B, C$, and $D$ are independent events (using the (1) definition above). Show that

$$\mathbb{P}(A \cap (B \cup C \cup D)) \ = \ \mathbb{P}(A)\mathbb{P}(B \cup C \cup D).$$

2. Show that the map

$$\varphi \; : \; \mathbb{N} \times \mathbb{N} \; \rightarrow \; \mathbb{N}$$

which sends

$$(a,b) \;\; \rightarrow \;\; a + \frac{(a+b-2)(a+b-1)}{2},$$

is a bijection. Here, $\mathbb{N}$ is the set of positive integers (i.e. Natural numbers). Hint: Draw the $5 \times 5$ square of numbers $(a,b)$, $1 \leq a, b \leq 5$, and see where $(a,b)$ gets sent.

3. In a certain "pick 3" lottery a person selects a number from among 1000 possibilities $\{000, ..., 999\}$. Let us suppose each number is equally likely. Now suppose 500 people play the game, and each picks a number. From the lottery commission's perspective, the worst thing that could happen is if the 500 people conspire and each pics a different number. If they do this, then the chance that at least one of them wins (and splits his winnings with the other 499 people he colludes with) is obviously 1/2. Now suppose the 500 people *don't* conspire, and each picks his or her number independently of the other players. Show that the probability that at least one of the players picks the winning number is approximately $1 - 1/\sqrt{e}$. In working this problem, state what your sample space is, what the probability measure is, and any assumptions you make.