Notes 2 for Honors Probability and Statistics

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1 Examples of σ-algebras and Probability Measures

So far, the only examples of σ-algebras we have seen are ones where the sample space is finite. Let us begin with a FALSE example of a σ-algebra where S is infinite:

False example of a σ-algbra. We say that a subset $A \subseteq \mathbb{N}$ has density ρ if and only if

$$\lim_{n \to \infty} \frac{\#\{a \in A\ :\ a \leq n\}}{n}\ =\ \rho.$$

Another, more compact way of writing the numerator in the limit is $|A(n)|$, which is the cardinality of the set $A(n)$, which in turn is the number of elements of A that are $\leq n$.

Not every subset of $\mathbb{N}$ has a density. For example, the set B with the property that $b \in B$ if and only if b is an integer lying in an interval $[2^{3j}, 2^{3j+1}]$ for some integer $j \geq 0$, does not have a density. Another way of writing B is as the infinite union

$$B\ =\ \{1,2\} \cup \{8, 9, ..., 16\} \cup \{64, 65, ..., 128\} \cup \cdots$$

To see that B does not have a density, consider the size of $|B(2^{3k} - 1)|$, for some integer $k \geq 1$. This set will have

$$(1+1) + (8+1) + (64+1) + \cdots + (2^{3(k-1)} + 1)\ =\ k + \frac{2^{3k} - 1}{7}\ \ \text{elements.}$$

So, $|B(2^{3k}-1)|/(2^{3k}-1)$ will tend to the limit $1/7$ as $k \to \infty$. But now consider the size of $B(2^{3k+1})$. This set is exactly the same as $B(2^{3(k+1)}-1)$, and so has size $(k+1)+(2^{3(k+1)}-1)/7$ elements; and so, $|B(2^{3k+1})|/(2^{3k+1})$ tends to the limit $4/7$ as $k \to \infty$. So, the limit $|B(n)|/n$ does not exist, meaning that B does not have a density.

Now, it seems intuitively obvious that if we let $S = \mathbb{N}$, and let Σ be the set of all subsets $A \subset S$ that *have* a density ρ, and then let $P(A) = \rho$, then (S, Σ, P) is a probability space. It turns out that this is not true! The problem here is that Σ is NOT a σ-algebra, and, in particular, is not closed under intersections. An example of a pair of sets $C, D \in \Sigma$ whose intersection is not in Σ is given as follows: Let C denote the set of even integers, and let $D = D_0 \cup D_1$, where D_0 is the set of even integers lying in an intervals of the form $[2^{3j}, 2^{3j+1}]$ (where $j \geq 0$ is an integer), and where D_1 is the set of odd integers that lie in intervals of the form $(2^{3j+1}, 2^{3(j+1)})$ (again, $j \geq 0$ is an integer). Then one can show C and D both have density $1/2$, but their intersection is the set of even integers in B, the pathelogical set we constructed above; and, the set of even integers in B likewise does not have a density. So, $C \cap D$ does not belong to Σ, which proves Σ is not a σ-algebra.

A positive example: Borel sets. Given a collection of sets C, which are to be subsets of some more basic set S (or Ω as we've said) we say that Σ is the σ-algebra generated by C, and we write it by $\Sigma = \sigma(C)$, if Σ is gotten by applying countable set operations (union and complementation) to C. Another definition for Σ is that it is the intersection of all the σ-algebras Σ_1 such that $C \subseteq \Sigma_1$. Here is an example: Start with $S = \{1,2,3,4\}$, and let $C = \{\{1,2\},\{3,4\}\}$. Then, it turns out that

$$\Sigma \;=\; \sigma(C) \;=\; \{\emptyset,\ \{1,2\},\ \{3,4\},\ \{1,2,3,4\}\}.$$

To see this, first note that Σ is a σ-algebra on $S = \{1,2,3,4\}$: We have that $S \in \Sigma$, the complements of each of the elements (subsets of S) are in Σ, and it is easy to see that all unions lie in Σ. Also, it is obvious that any σ-algebra containing C must likewise contain Σ. So, Σ is the σ-algebra generated by C, i.e. it is the intersection of all σ-algebras containing C.

In this case Σ was not equal to the power set 2^S; however, had we chosen $C = \{\{1\},\{2\},\{3\},\{4\}\}$, then Σ *would* have been the powerset 2^S.

Given a subset X of the real numbers, and given a point $x \in X$, we say that the set

$$\{y \in X \ : \ |x - y| < \epsilon\}$$

is an ϵ-neighborhood of x relative to X, and we denote the set of points y by $N_X(x;\epsilon)$. We say that a subset $Y \subseteq X$ is *open relative to* X if and only if for every point $y \in Y$ there is a neighborhood $N_X(y;\epsilon)$ contained within Y; that is, $N_X(y;\epsilon) \subseteq Y$. We also say that a subset Y is closed relative to X if its complement $\overline{Y} = \{y \in X \ : \ y \notin Y\}$ is open. Note that the empty set [1] and X itself are both open *and* closed relative to X. This is probably not the definition of open sets you are used to, so let us look at a few examples, to make sure you understand it:

Example 1: Let X be the closed interval $[0,1]$. Then, the half open interval $[0,1/2)$ is a 1/2-neighborhood of 0 relative to X, since

$$[0,1/2) \ = \ \{y \in [0,1] \ : \ |0 - y| < 1/2\};$$

however, it is NOT a 1/2-neighborhood of 0 relative to $\mathbb{R}$. I also claim that $[0,1/2)$ is an open set relative to X.

Example 2: Let $X = \{1,2,3,4\}$. Then, the single point 3 is a 1-neighborhood of 3 relative to X, since

$$\{3\} \ = \ \{y \in X \ : \ |3 - y| < 1\}.$$

In fact, the set of all the neighborhoods relative to X are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{1,2,3\}$, $\{2,3,4\}$, and $\{1,2,3,4\}$. In this example, all the subsets of X are both open and closed relative to X, as you can check.

Now we can define what the Borel sets of $\mathbb{R}$ are: Given a subset $X \subseteq \mathbb{R}$, let U be the set of subsets of X that are open relative to X; that is,

$$U \ = \ \{Y \subseteq X \ : \ Y \text{ open relative to } X\}.$$

Let $B(X) = \sigma(U)$ be the σ-algebra generated by these open subsets U. The B here stands for A. Borel. $B(X)$ is called the "Borel σ-algebra generated by X", which means that "$B(X)$ is the σ-algebra generated by the open subsets of X".

[1] The empty set is vacuously an open set relative to X.

Let us now consider what types of subsets are contained in $B(\mathbb{R})$. We first claim that $B(\mathbb{R})$ is the σ-algebra generated by all the open intervals (a, b), where $a < b$; that is,

$$B(\mathbb{R}) \;=\; \sigma(\{(a,b) \;:\; a < b\}). \tag{1}$$

Basically, to prove this fact, one just needs to show the following basic fact that I assigned for homework:

Claim. Every open subset of $\mathbb{R}$ is a countable union of open intervals (a, b).

This claim implies that if U is the set of open subsets of $\mathbb{R}$, then

$$U \subseteq \sigma(\{(a,b) \;:\; a < b\}) \;\subseteq\; \sigma(U),$$

which in turn implies

$$B(\mathbb{R}) \;=\; \sigma(U) \;\subseteq\; \sigma(\{(a,b) \;:\; a < b\}) \;\subseteq\; \sigma(U),$$

which proves (1). The fact we've used here is that if $A \subseteq \sigma(B)$, then $\sigma(A) \subseteq \sigma(B)$.

As promised, here are some subsets of $B(\mathbb{R})$:

I. Single points $\{a\} \in B(\mathbb{R})$. To see this, consider the open intervals $I_n = (a - 1/n, a + 1/n)$ for $n \in \mathbb{Z}^+$. Then,

$$\{a\} \;=\; \bigcap_{n=1}^{\infty} I_n,$$

and note that this intersection lies in $B(\mathbb{R})$, being a countable intersection on open intervals.

II. Any countable subset of the reals $A = \{a_1, a_2, ...\}$ lies in $B(\mathbb{R})$. This is because A is a countable union of elements of $B(\mathbb{R})$, namely the singleton sets $\{a_n\} \in B(\mathbb{R})$.

III. Any closed or half closed interval lies in $B(\mathbb{R})$, since, for example,

$$[a,b] \;=\; (a,b) \cup \{a\} \cup \{b\},$$

is the union of three elements of $B(\mathbb{R})$.

IV. Any half-infinite interval lies in $B(\mathbb{R})$. For example,

$$(-\infty, a) = \bigcup_{j=1}^{\infty}(a - j, a - j + 1).$$

Also, the intervals $(-\infty, a]$, (a, ∞) and $[a, \infty)$ all lie in $B(\mathbb{R})$.

V. Any countable union of intervals, finite or infinite, open, closed, or half open or closed, lie in $B(\mathbb{R})$.

So you see, $B(\mathbb{R})$ is *extremely* general, and will certainly be large enough for all the examples we will do in this course (we may need to use $B(\mathbb{R}^n)$, and I hope you can surmise its definition).

1.1 Probability Measures on Borel Sets

So far the only type of measure on a σ-algebra we have studied is a probability measure. A probability measure is actually a special case of a measure: A measure μ on a σ-algebra Σ is a mapping $\mu : \Sigma \to [0, \infty]$ (note here that we include ∞ in the arrival space; there may be some subsets in Σ with infinite μ-measure) which is countably additive, by which we mean that if $A_1, A_2, ...$ is a contable sequence of disjoint subsets of a set S, where each A_i lies in Σ, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_i \mu(A_i).$$

The σ-algebra $B(\mathbb{R})$ has a standard measure μ, which is called the Borel measure (for obvious reasons!), and it assigns the intervals (a, b) the value $b - a$; that is, $\mu((a, b)) = b - a \in [0, \infty]$. Thus, this measure is consistent with our usual notion that the "length" or "measure" of the interval (a, b) is $b - a$. However, *proving* there exists a measure μ on all of $B(\mathbb{R})$ having this property is not easy. The main problem is showing that μ is well-defined; that is, there are many different ways of building up a subset of $\mathbb{R}$ by applying set operations to intervals (a, b), and the value of μ applied to all these various constructions for the same subset has to be the same. The main tool used to prove that there is a measure μ on $B(\mathbb{R})$ consistent with our usual definition of length (so, $\mu((a, b)) = b - a$) is the following theorem due to Caratheodory:

Theorem (Carathéodory's Extension Theorem). Let S be a set, and let Σ_0 be an algebra on S; that is, Σ_0 is a collection of subsets of S satisfying the following properties:

1. $S \in \Sigma_0$;
2. If $A \in \Sigma_0$, then $\overline{A} \in \Sigma_0$; and,
3. If $A, B \in \Sigma_0$, then $A \cup B \in \Sigma_0$.

Suppose that $\mu_0 : \Sigma_0 \to [0, \infty]$ is a countably additive map on Σ_0. Then, if we let Σ be the σ-algebra generated from Σ_0, we have that there exists a measure μ on Σ which agrees with μ_0 on Σ_0; that is, if $C \in \Sigma_0$, then $\mu(C) = \mu_0(C)$. Furthermore, if μ_0 is bounded, then μ is unique.