

# Some practice problems involving Markov Chains

December 20, 2010

1. Show that if  $T$  is a Markov chain such that there is a positive probability of transitioning from any vertex to any other vertex in a single time step (note: we are not saying that there is a positive probability of transitioning from vertex  $i$  to itself in one time step), then  $T$  has a unique equilibrium distribution.
2. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become well. Find the steady state for the corresponding Markov process.
3. Suppose that an ant starts at position  $(0, 0)$ , and can move up, down, left, or right one unit, so long as the new position stays inside the grid  $\{(x, y) : 0 \leq x, y \leq 2\}$ . Determine the probability that the ant reaches position  $(2, 2)$  within 20 moves.
4. Find an example of a connected Markov chain – you can reach any state from any other state (ignoring edge directions) – that has more than one equilibrium distribution.
5. Population dynamics can often be modeled by composing polynomials, and in this problem we will explore this possibility: suppose that you have a population of bacteria whereby from one generation to the next, a single individual will produce  $j$  offspring with probability  $p_j$ , and that that individual dies in the next generation. Let

$$f(x) = \sum_{j=0}^{\infty} p_j x^j.$$

Say that in our case

$$f(x) = 1/2 + 1/4x + 1/4x^2.$$

If at time 0 the population has 1 individual, determine the expected number of individuals for when time  $t = 4$ . Hint: think about what  $f(f(x))$  means, and then  $f(f(f(x)))$ , etc.