Notes on Poisson Processes

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1 Introduction

Depending on the book (or website) you read, a "Poisson Process" can have many different definitions. For me, the key axioms defining it are as follows: First, we fix a time interval $[0, T]$, and a certain parameter λ, and we have associated to this interval a certain number X of events that can occur, satisfying:

- Associated to any set of DISJOINT subintervals $I_1, ..., I_k \subseteq [0, T]$, we have INDEPENDENT random variables $X_{I_1}, ..., X_{I_k}$, where X_{I_j} is the number of events occurring in the time window I_j.

- Let $I := [x, x + h] \subseteq [0, T]$. Then,

$$\lim_{h \to 0} \frac{\mathbb{P}(X_I = 1)}{\lambda h} \ = \ 1.$$

  That is to say, as h tends to 0, $\mathbb{P}(X_I = 1)$ grows like λh.

- Using the same interval I as in the above, we have that the probability that 2 or more events occur in I has size $o(h)$; that is,

$$\lim_{h \to 0} \frac{\mathbb{P}(X_I \geq 2)}{h} \ = \ 0.$$

In the third item we used little-oh notation $o(h)$. Let us remind ourselves what this means, since we will use it later throughout the course: Given positive functions $f(x)$ and $g(x)$, we say that $f(x) = O(g(x))$ if there exists a constant $c > 0$ such that

$$f(x) \ < \ cg(x) \tag{1}$$

for sufficiently large values of x (say, $x > x_0$, for some x_0). And we say that $f(x) = o(g(x))$ if for every $c > 0$ there exists $x_0(c)$ such that

$$f(x) \ < \ cg(x). \tag{2}$$

In other words, $f(x)$ grows slower than any fixed positive constant multiple of $g(x)$ once x is large enough. We could alternatively say here that $f(x) = o(g(x))$ means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \ = \ 0.$$

In the above usage of little-oh notation (in defining a Poisson process), note that we take $h \to 0$, not ∞. Well, the idea for how to define little-oh and big-oh is much the same for this case: We say that $f(x) = O(g(x))$ as $x \to 0$ if there exists some constant $c > 0$ such that (1) holds for all x sufficiently close to 0; and we say that $f(x) = o(g(x))$ as $x \to 0$ if for every constant $c > 0$ there exists $x_0(c) > 0$ such that (2) holds for $0 < x < x_0(c)$. We also can use the limit definition here; that is, $f(x) = o(g(x))$ as $x \to 0$ if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} \ = \ 0.$$

2 Poisson processes lead to Poisson distributions

It turns out that a random variable X determined via a Poisson process as in the previous section, has a Poisson distribution; and basically this will follows from a combination of several ideas we have seen previously, including facts about the binomial distribution, the union bound, and independence.

For convenience we set $T = 1$ and $h = 1/n$, and then later we will let $n \to \infty$. Define the random variables

$$X_1 \ := \ X_{[0,h)}, \ X_2 \ := \ X_{[1,2h)}, \ ..., \ X_n \ := \ X_{[(n-1)h,1]}.$$

Then, the total number of events $X := X_{[0,1]}$ satisfies

$$X = X_1 + \cdots + X_n.$$

Now, for h small enough these X_i's are *essentially* Bernoulli random variables; and so, X is then *essentially* a binomial r.v. But we have to deal with the cases where $X_i \geq 2$: Define

$$E := (X_1 \geq 2) \cup (X_2 \geq 2) \cup \cdots \cup (X_n \geq 2).$$

Although the X_i's are *independent*, we do not have that these events here are *disjoint*; however, from the union bound we know that

$$\mathbb{P}(E) \leq \mathbb{P}(X_1 \geq 2) + \cdots + \mathbb{P}(X_n \geq 2) = no(h) = o(1).$$

So, the larger we take n, the closer to 0 we will get $\mathbb{P}(E)$ to be.

Consider now the event $X = j$. This can either occur by having the X_i's take on the values 0 and 1 only, resulting in $\binom{n}{j}$ ways of summing to j; or can occur when some of the $X_j \geq 2$. As we have already said, the latter case accounts for essentially 0 probability as $n \to \infty$. So we only need to consider the case where the X_i's are 0 or 1; and given that we are in this case, we must have that

$$\mathbb{P}(X_i = \delta) \sim \begin{cases} \lambda/n, & \text{if } \delta = 1; \\ 1 - \lambda/n, & \text{if } \delta = 0. \end{cases}$$

It follows, then, that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(X = j) &= \lim_{n\to\infty} \binom{n}{j} (\lambda/n)^j (1 - \lambda/n)^{n-j} \\ &= (\lambda^j/j!) \lim_{n\to\infty} \frac{n(n-1)\cdots(n-j+1)}{n^j} (1-\lambda/n)^n (1-\lambda/n)^{-j}. \end{aligned}$$

Clearly, as $n \to \infty$ we have that the $n(n-1)\cdots(n-j+1)/n^j \to 1$, as does $(1-\lambda/n)^{-j}$, while the remaining factor $(1-\lambda/n)^n$ tends to $e^{-\lambda}$. So, in the limit as $n \to \infty$ we have

$$\mathbb{P}(X = j) = \lambda^j e^{-\lambda}/j!,$$

which means that X has a Poisson distribution.