Math 3225 Final Exam, Fall 2005

December 13, 2010

1. Define the following

a. A sigma algebra.
b. The Kolgomorov axioms of probability.
c. The Law of Large Numbers.
d. The Central Limit Theorem.

2. Bob wishes to transmit one bit of information across a noisy channel to Alice. If the bit to be transmitted is '1', then Bob sends three 1's in a row; and, if the bit to be transmitted is '0', then Bob sends three 0's. Alice's rule for determining which bit Bob wanted her to receive is 'Majority Rules': If the number of 1's among the three received bits is 2 or 3, then Alice reports that Bob was trying to send her a '1', and if the number of 0's among the three received bits is 2 or 3, then Alice reports that Bob was trying to send her a '0'.

Suppose that the channel makes an error $1/3$ of the time (thus, if you sent 1000 1's in a row, about 333 of them will be flipped to 0's, and whether a bit is flipped is independent of whether the other bits get flipped). And suppose that Bob chooses the bit 0 or 1 of information with equal probability. If Alice recieves the noisy message '110', what is the probability that Bob's intended bit was, indeed, '1', as Alice reports?

3. A pick 3 lottery has 1000 possible numbers, 000 through 999. Suppose that 1000 people play the lottery, and each picks their number uniformly from among all possibilities and independently of the other numbers chosen by the other players. Prove that the probability that some player wins is approximately $1 - 1/e$. (If all players collude, and each picks a different number, then clearly there will be a sure winner. What this means is that if the lottery commission knows that players won't collude, they can afford to offer larger prizes than would seem possible if they wish to generate a profit.)

4. Let Σ be the usual Borel sigma-algebra on $[0,1]$, and let P be the Borel probability measure on Σ, which assigns the interval $[a,b] \subseteq [0,1]$ the value $P([a,b]) = b - a$. Let I denote the set of irrational numbers belonging to $[0,1]$. Determine $P(I)$. Explain your answer, and quote any theorems that you use.

5. You have a pair of ants. The first ant is located at position $X_0 = 0$ at time $t = 0$, and the second ant is located at position $Y_0 = 10$ at time $t = 0$. Given the position X_t and Y_t of the first and second ant at time t, we assume each ant moves to the right by one unt or to the left by one unit with equal probability, and the movement each ant takes is independent of the movement of the other ant at all times and independent of its own previous movements. That is, $P(X_{t+1} = X_t + 1 | X_0, ..., X_t, Y_0, ..., Y_{t+1}) = P(X_{t+1} = X_t + 1) = 1/2$.

Determine $P(X_{10} = Y_{10})$.

6. You read a newspaper article which says that 30 percent of the population believe in UFO's. You then decide to test this by conducting a poll of 10000 people (without replacement). Let X be the number of people in the random sample who believe in UFO's. Assume that X is approximately normally distributed as prediced by the Central Limit Theorem.

Suppose that in your random sample you find that 3100 people believe. To test whether to reject your hypothesis you compute

$$P_{30}(|X - 3000| \geq 100),$$

which is the probability that $|X - 3000| \geq 100$, given that 30% of the population believe in UFO's. Find this probability.

7. Prove that if X_1, X_2, X_3 are random variables that are pairwise independent (which means that the pairs X_1, X_2 and X_1, X_3 and X_2, X_3 are independent, yet X_1, X_2, X_3 all three may not be independent), then $V(X_1 + X_2 + X_3) = V(X_1) + V(X_2) + V(X_3)$. Recall that we know this also holds under the tighter condition that V_1, V_2, V_3 are all independent (not pairwise independent).

8. A large population of rats are put into a maze, and after traveling, wind up at position A or position B. If a rat starts at position A at time t there is an 85% chance it will remain in position A at time $t+1$, and if it starts in position B at time t there is a 90% chance it will remain in position B at time $t+1$. Assuming that the population of rats that wind up in position A or B evolves according to the rules of a markov process with these transition probabilities, determine the equilibrium population.

9. Suppose that $\{X_t : t \geq 0\}$ is a standard Brownian motion process with drift μt; so, $X_t = B(t) + \mu t$, where $B(t)$ is the standard Brownian motion process. We wish to determine a maximum liklihood estimate for μ – call this estimate $\hat{\mu}$ – by making a series of observations of X_t at times $t_1, t_2, ..., t_k$. Say the observations yield $X_1 = x_1, ..., X_k = x_k$. Determine $\hat{\mu}$ as a function of $x_1, ..., x_k$ and (possibly) $t_1, ..., t_k$.

10.

a. Determine the moment generating function of the exponential random variable X whose pdf is $f(x) = 3e^{-3x}$.

b. Using your answer from part a, determine the sixth moment of X, which is $\mathbb{E}(X^6)$.