A_n is generated by the 3-cycles

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Here I give a proof of one of your homeworks on A_n being generated by the 3-cycles.

The idea of the proof is that we will show that any product of two transpositions is a product of 3-cycles; and, if so, then since every element of A_n is a product of an even number of transpositions, we will be done.

To begin, let us suppose that we have the product of transpositions $\tau_1 \tau_2$, where τ_1 and τ_2 both move a common number $a \in \{1, ..., n\}$ – say they both move a. Then, we will have that τ_1 and τ_2 must have the form

$$\tau_1 \; = \; (a \; b), \; \tau_2 \; = \; (a \; c).$$

In this case, we have

$$\tau_1 \tau_2 \; = \; (a \; b)(a \; c) \; = \; (a \; c \; b),$$

and we are done.

Now suppose that τ_1 and τ_2 move different numbers; then, we may write

$$\tau_1 \; = \; (a \; b), \; \tau_2 \; = \; (c \; d),$$

where a, b, c, d are all different numbers. Then, we have that

$$\tau_1 \tau_2 \; = \; (d \; a \; c)(a \; b \; d).$$

So, we again have that the product of transpositions is a product of 3-cycles. We are now done.