Selected solutions to Math 4107 midterm 2, Fall 2009

December 11, 2009

1. You can look these up.
2. This is a routine computation, so I won't bother to do it.
3. The center of S_2 is all of S_2, since S_2 is cyclic of order 2, which is abelian.

We claim that the center of S_n for $n \geq 3$ is just $\{e\}$, the identity: To see this, suppose φ is an element of the center. If φ is not the identity, then there exists $a, b \in \{1, 2, ..., n\}$, $a \neq b$, such that

$$\varphi(a) \;=\; b.$$

So, if we let $c \in \{1, 2, ..., n\}$ be distinct from a and b (possible, since $n \geq 3$), we have that

$$\varphi \circ (a\ c)\ \circ \varphi^{-1} \;=\; (b\ \varphi(c)) \;\neq\; (a\ c).$$

This contradicts the fact that φ was in the center; and so, the center contains only the identity.

4. If $|G| = pq$, $q > p$, then G has Sylow-p and Sylow-q subgroups of orders p and q, respectively. But how many Sylow subgroups? The number of Sylow-q's is $1, q+1, 2q+1, ...$ But it can't be $2q+1$ or higher, since if any pair of Sylow-q's shared a non-idenitty element they would have to be the same subgroup; so, if there were $2q+1$ or more of them, then they would, collectively, contribute at least

$$(2q+1)(q-1) \;>\; pq \;=\; |G|$$

elements of order q, impossible. So, there is only one Sylow-q, and therefore it must be normal.

The number of Sylow-p's is $1, p+1, 2p+1, \ldots$. And, this number must divide $|G|$ (why? Because the Sylow-p's are all conjugate to one another, meaning they form an orbit under conjugation; by Orbit-Stabilizer, this orbit size divides $|G|$). But since $kp+1$ is coprime to p, we must therefore have $(kp+1)|q$. So, in fact, either $k=0$ or $kp+1=q$, since q is prime. From our assumption that p does not divide $q-1$, we conclude that this is impossible for $k \geq 1$ and $kp+1=q$, meaning that $k=0$; in other words, the Sylow-p is normal.

So, we have that $G = P_p P_q$, the direct product of its normal Sylow subgroups (which only intersect in the identity), and we know that this is isomorphic to $P_p \times P_q$, which is abelian (in fact, it is cyclic of order pq), being the cross product of two cyclic groups.

5. This is a routine computation, so I won't bother to do it.