Final Exam, Math 4107

December 12, 2005

1. (Easy) Define the following terms:
a. Group
b. Ring
c. Field
d. Principal Ideal Domain
e. Euclidean Domain

2. (Medium) In this problem you will construct a non-commutative group of order 21: We will take the set G defining our group to be $\mathbb{Z}_3 \times \mathbb{Z}_7$, and we will define a funny addition. This addition works by

$$(a, b) + (c, d) \;=\; (a + c \pmod 3,\; b \cdot 2^c + d \pmod 7).$$

a. Prove that this is indeed a group. (Hint: Be careful in showing that the group operation is well-defined; also, associativity is slightly tricky.)

b. Prove that this group is non-commutative.

c. (Optional +1 point extra credit) Why doesn't this example work to show that there is a group of order 15 which is non-abelian? (We know all groups of order 15 are abelian).

I could explain in about half a class where this example comes from. It is an example of a semi-direct product of $\mathbb{Z}_3$ and $\mathbb{Z}_7$. Alas, the semester is over and our time has come to an end...

3. (Easy) Give an example of an additive abelian group G, and a triple of subgroups H_1, H_2, H_3 satisfying the following two conditions:
1. $H_i \cap H_j = \{0\}$ for all $1 \leq i < j \leq 3$.
2. $|H_1 + H_2 + H_3| \;\neq\; |H_1||H_2||H_3|$.

4. (Medium)

a. Prove that if aH and bH are pairs of cosets of a subgroup H of a group G that have an element in common, then $aH = bH$.

b. Using part a, prove Lagrange's theorem, which says that if $H < G$, then $|H|$ divides $|G|$.

5. (Medium/Easy) Suppose that G is a group of order $2 \cdot 3 \cdot 5 \cdot 7$, and suppose G has a normal subgroup of order 15. Prove that that sugroup of order 15 has a normal subgroup of order 5, and then prove that G itself has a normal subgroup of order 5.

6. (Easy) Let α be the following permuation belonging to S_{10}:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 4 & 10 & 5 & 2 & 8 & 3 & 6 & 9 & 1 \end{pmatrix}.$$

a. Express α as a product of disjoint cycles.

b. Determine whether α is even or odd.

c. Consider the subgroup $(\alpha) < S_{10}$ generated by taking powers of α. Determine the order of this cyclic subgroup.

7. (Medium) Suppose that φ is a surjective group homomorphism from a group G of order pn to an abelian group I of order n, where p is prime. If G has a subgroup H of order n, and if $p \not| n$, then show that H must be abelian.

8. (Medium) Give an example of each of the following, and provide explanations (or proofs as needed):

a. A ring R which is not a principal ideal domain. Provide proofs.

b. A non-trivial ring homomorphism $\varphi : R \to S$, where R and S are rings, such that $\varphi(1) \neq 1$.

c. A ring which is a unique factorization domain, but is not a Euclidean domain.

9. (Easy) An ideal P in a ring R is called a *prime ideal* if whever a product ab belongs to P we either have $a \in P$ or $b \in P$. Prove that if P is a prime ideal, then R/P is an integral domain.

10. (Medium) We defined a zero divisor d in a ring R to be any non-zero element for which there exists another non-zero element e such that $de = 0$

or $ed = 0$. In this problem we will see why it is cruical to include both possibilities $de = 0$ and $ed = 0$: Consider the ring R formed by taking the set of all 2×2 matrices with real coefficients.

a. Prove that this really is a ring, by showing that it satisfies the axioms of a ring, where the additive operation is just matrix addition, and the mutliplicative operation is matrix multiplication.

b. Produce a pair of matrices A, B in the ring such that $AB = 0$, but $BA \neq 0$.