Math 4107 final exam, Fall 2009

December 11, 2009

1. Define the following terms.
  a. Group
  b. Interal direct product.
  c. Fundamental theorem of finite abelian groups.
  d. Integral domain.
  e. Sylow theorems (list all three).

2.
  a. Write the following permutation in disjoint cycle notation (composition works from right-to-left):

$$(1\ 2\ 3\ 4\ 5)(1\ 3\ 7\ 2)(1\ 3\ 6\ 4)(2\ 1\ 3\ 5).$$

  b. Determine whether this permutation is even or odd, and explain your answer.

3. Determine **all** integers x and y satisfying

$$511x + 851y \;=\; 1,\; |x| \leq 850,\; |y| \leq 510.$$

4. Determine the number of non-isomorphic abelian groups of order $720 = 2^4 \cdot 3^2 \cdot 5$, and list one from each isomorphism class.

5. Let $d(a + bi) = a^2 + b^2$. Find $r, q \in \mathbb{Z}[i]$ such that

$$171 + 41i \;=\; (74 + 18i)q + r,\; \text{where } d(r) < d(74 + 18i).$$

6. Determine the conjugacy classes of D_6 (that is, the orbits under conjugation).

7.

a. Prove that if G is a group of order p^n, then G has a non-trivial center.

b. Using your answer from part a, and some induction, show that if G has order p^n, then G has a **normal** subgroup of order p^{n-1}.

8. In this problem we will construct a non-abelian group of order 27 using semi-direct products: Basically, find θ making $\mathbb{Z}_9 \rtimes_\theta \mathbb{Z}_3$ into a non-abelian group of order 27. Explain your work.

9. Suppose R is a commutative ring and S is an integral domain. Prove that if there exists an injective homomorphism $\varphi : R \to S$ (which we don't assume is necessarily surjective), then R must be an integral domain as well.

10. Prove that the following is a group: Let G be the set of all mappings $f : \mathbb{C}^* \to \mathbb{C}^*$ ($\mathbb{C}^*$ refers to the extended complex plane, and is just the usual complex numbers, together with the point at infinity) of the form

$$f(\tau) = \frac{a\tau + b}{c\tau + d}, \text{ where } a, b, c, d \in \mathbb{R}, \text{ and } ad - bc \neq 0,$$

where the operation for G is just composition of functions (i.e. 'multiplying' f and g in the group amounts to computing $f \circ g$).