Math 4107, Midterm 2

April 3, 2008

1. (easy) Determine the number of non-isomorphic abelian groups of order 72, and list one group from each isomorphism class.

2.(easy)

a. Express the following permutation in S_9 in disjoint cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 2 & 1 & 3 & 9 & 8 & 4 & 6 \end{pmatrix}.$$

b. Determine whether the permutation in part a is even or odd.

3.(easy) Define/explain the following (as we have used them in the course):

a. Ring

b. Fundamental Theorem of Finite Abelian Groups.

4.(somewhat easy) Recall that all groups of order p^2, p prime, are abelian. It is not true that all groups of order p^3 are abelian, however, and there is a standard construction of this. In this exercise you will verify this construction.

a. Consider the set H consisting of all 3×3 matrices A of the following form

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where all the entries lie in the ring $\mathbb{Z}_p$, the integers modulo p. There are clearly p^3 such matrices (there are p choices for each entry). These matrices lie in a much larger group G consisting of all 3×3 matrices with entries in the ring $\mathbb{Z}_p$ whose determinants are not divisible by p. The natural operation in G is matrix multiplication (thus, the group operation is: You multiply two matrices together in the usual way, and then you mod out by p). Assuming that G is indeed a group under matrix multiplication, prove that H is a subgroup of G under matrix multiplication (recall the one-step subgroup test for finite groups).

b. Verify that H is non-abelian.

5.(medium) Show that every group G of order 56 is non-simple (that is, every group of order 56 has a normal subgroup H, where $H \neq G$ and $H \neq \{e\}$). Here you are advised to do this in three steps:

a. Use Sylow's theorem to show that there are either 1 or 8 Sylow-7 subgroups.

b. If G has 8 Sylow-7 subgroups, show that it must have exactly 1 Sylow-2 subgroup by a careful accounting of the number of elements in G having order 7.

c. Conclude G is non-simple.