Selected Solutions to Math 4107, Set 1

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11. a. What is wrong is that you may have that a is not related to anything at all, and so you cannot complete the proof

$$a \sim b \overset{\text{reflexive}}{\Longrightarrow} b \sim a \overset{\text{transitive}}{\Longrightarrow} a \sim a,$$

because you fail at the start.

b. If you replaced the reflexive property with the property "Every element a of your set is related to *something* (perhaps only to a itself)".

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4. If $a|x$, then $x = ak$, for some integer k. Then, if $b|x$ we deduce that $b|ak$. But, since $(b, a) = 1$ we conclude from Lemma 1.3.2 that $b|k$. Thus, $k = b\ell$, which gives $x = ak = ab\ell$, and therefore $ab|x$.

Another way you can prove this is to use the fundamental theorem of arithmetic, and the basic fact that $u|v$ if and only if for every prime p we have $u_p \leq v_p$, where u_p and v_p are the powers of p dividing u and v, respectively. First, let us prove this basic fact: Define

$$k \ = \prod_{\substack{p|u, \text{ or } p|v \\ p \text{ prime}}} p^{v_p - u_p}.$$

Since $v_p \geq u_p$ for all primes p we deduce that k is a positive integer. But now, $u_p k = v_p$, which implies $u_p | v_p$.

This solution to our problem 23 now goes as follows: Let a_p, b_p, x_p denote the exact power of p that divides a, b, and x, respectively; that is, $p^{a_p}||a$, $p^{b_p}||b$, and $p^{x_p}||x$. If $a|x$ we have that $a_p \leq x_p$ for all primes p, and if $b|x$ then $b_p \leq x_p$ for every prime p. So, we deduce that if a, b both divide x, then

$$\max(a_p, b_p) \ \leq \ x_p,$$

for all primes p.

Now, the condition $(a, b) = 1$ means that for every p either $a_p = 0$ or $b_p = 0$, which gives

$$a_p + b_p \;=\; \max(a_p, b_p) \;\leq\; x_p.$$

As $p^{a_p+b_p} || ab$, it now follows from our above observation (concerning u_p and v_p) that $ab|x$.