Final Exam Study Sheet

April 30, 2010

- You will have 10 questions, and 3 hours to complete the exam. If you score is higher on the final exam than you score up to and including the final, then your course grade will be the final exam grade (maximal grading policy).

- First, look at the study sheets for midterm 1 and midterm 2. In addition to *that* material, you will be expected to know the materials below.

- The "new stuff" is basically some of the material on tensors of a ring with a module, and associate algebras.

- Know what is meant by $S \otimes_R M$, where R is a subring of a ring S. Know what is meant by the "rank of an R-module", where R is a PID (actually, you can also do it, more generally, for R an integral domain). Basically, using the Fundamental Theorem of Finitely Generated Modules over a PID, we know that

  $$M \;=\; M_1 \oplus M_2 \oplus \cdots \oplus M_k,$$

  where the M_i are cyclic R-modules (this is a direct sum here).

  Now, some of these cyclic modules will be finite, and some will be infinite. Collect the finite ones into a single module M_{Tor}, where Tor refers to "torsion" (an element g of an algebraic structure is called "torsional" if there is some 'multiple' of the element that equals the 0 element); and, M_{Free} denotes the other cyclic modules. We will have then that

  $$M \;\cong\; M_{\text{Tor}} \oplus M_{\text{Free}};$$

and

$$M_{\text{Free}} \;\cong\; R \times R \times \cdots \times R \;=\; R^r.$$

This r is the "rank" of the module.

Now, it will turn out that if S is the field of fractions of R, then $S \otimes_R M$ is a vector space, isomorphic to the vector space S^r. And, we then know from linear algebra that the dimension of a vector space is well-defined, so r is well-defined both for this vector space as well as for the module.

- Know the definition of an associative R-algebra. Know how to produce one from an R-module with an extra multiplication rule; and, know how to produce one from a ring S, together with a homomorphism $\varphi : S \to R$. Know how to also produce the ring (with homomorphism) and module, given the algebra. Know some examples of associative algebras: First, every ring R can be considered as a $\mathbb{Z}$-algebra in an obvious way; the ring of $n \times n$ matrices with real coefficients forms an associative algebra; also, the "endomorphism ring" for an abelian group is a $\mathbb{Z}$-algebra. Know that "cross products" does not form an associative algebra.