Solutions to Homework 4

May 1, 2010

1. We will prove this via induction: The first term in the Upper Central Series is $\{e\}$, which is clearly a characteristic group. Suppose, for proof by mathematical induction, we have shown that

$$\{e\} = Z_0 \ \leq Z_1 \ \leq \ \cdots \ \leq \ Z_k$$

are all characteristic subgroups of G.

Let us now consider Z_{k+1}: Let

$$\psi \ : \ G \ \rightarrow \ G/Z_k$$

be the obvious mapping (note that Z_k is normal in G). Notice that since Z_k is characteristic, we have that σ will map the kernel of ψ into itself; in other words,

$$\psi(x) = e \quad \Longleftrightarrow \quad (\psi\sigma)(x) = e. \tag{1}$$

Let

$$Z' \ = \ Z(G/Z_k)$$

Then, Z' is normal in G/Z_k, and its inverse image via ψ is normal in G, and is what we call Z_{k+1}.

Now suppose $z \in Z_{k+1}$. We will now attempt to show that $(\psi\sigma)(z)$ commutes with everything in G/Z_k, which would prove $(\psi\sigma)(z) \in Z'$, and therefore $\sigma(z) \in Z_{k+1}$; applying the same argument using σ^{-1}, which is also an automorphism of G, we will get $\sigma(Z_{k+1}) = Z_{k+1}$, thereby completing the proof of the induction step.

Basically, we will show this by letting $a \in G/Z_k$ be arbitrary, and then noting that there exists $b \in G$ such that

$$(\psi\sigma)(b) \ = \ a.$$

Then, to prove $(\psi\sigma)(z) \in Z'$ we compute the commutator

$$[(\psi\sigma)(z),\ (\psi\sigma)(b)] \ = \ (\psi\sigma)(z^{-1}b^{-1}zb). \tag{2}$$

Now, since ψ maps $z^{-1}b^{-1}zb$ to e, by (1) we have the same is true for $\psi\sigma$, and so we are done, since the commutator (2) is e and b was arbitrary.

2. First, note that every Sylow-p subgroup of order p^n of a group G *itself* has subgroups of all orders p^j, $j = 1, 2, ..., n$; and, since

$$G \ \cong \bigoplus_{\substack{p_i || G| \\ p_i \text{ prime}}} P_{p_i},$$

where P_{p_i} is the Sylow-p_i subgroup, we can "build" a subgroup H of any given order, by extracting subgroups of the appropriate order $p_i^{a_i} \| |H|$ within the appropriate Sylow subgroups P_{p_i}.

For the converse, let us assume that G has a *normal* subgroup of each order (I should have probably added that assumption) dividing $|G|$. Then, in particular this implies that the Sylow subgroups are all normal, and it is then an easy exercise to see that this implies that G is the internal direct product of its Sylow subgroups (first, from normality we get that the product of Sylow subgroups is itself a subgroup; and then the unique representation part follows from the fact that the intersection of each Sylow subgroup with the product of the other Sylows, must be trivial). We are now done, as this is one of the characterizations of a nilpotent group.

3. You can do this yourself – it is a simple calculation.