A Curious Identity

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Here we consider
$$\sum_{n=2}^{\infty} \frac{n^{n-2}}{n!e^n} \ = ???.$$

First, we observe that
$$\frac{n^{n-2}}{n!e^n} \ = \ \text{Coef. of} \ \ x^{-1} \ \text{in} \ \frac{\exp(nx)}{n^2 e^n x^{n+1}}.$$
Now, this coefficient can be found using contour methods. In particular, by Cauchy's formula, we have that this coefficient is
$$\frac{1}{2\pi i e^n n^2} \int_{\ell_1+\ell_2+\ell_3+\ell_4} \frac{\exp(ns)}{s^{n+1}} ds,$$
where ℓ_1 is a vertical line stretching from $1-iN$ to $1+iN$, ℓ_2 stretches from $1+iN$ to $-M+iN$, ℓ_3 stretches from $-M+iN$ to $-M-iN$, and ℓ_4 stretches from $-M-iN$ to $1-iN$.

Now, for fixed $M>0$, letting $N \to +\infty$ we see that the contribution of the contours ℓ_2 and ℓ_4 becomes negligable; and so, our integral is equal to the integral over ℓ_1 and ℓ_3, with N replaced by $+\infty$. Likewise, by letting M tend to $+\infty$, one sees that the contribution to the integral along ℓ_3 becomes negligable. Thus,
$$\frac{n^{n-2}}{e^n n!} \ = \ \frac{1}{2\pi i n^2 e^n} \int_{1-i\infty}^{1+i\infty} \frac{\exp(ns)}{s^{n+1}} ds \ = \ \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} \frac{\exp(int)}{(1+it)^{n+1}} dt.$$
Summing over all $n \geq 2$, we find that
$$\sum_{n=2}^{\infty} \frac{n^{n-2}}{n!e^n} \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=2}^{\infty} \frac{\exp(int)}{n^2(1+it)^{n+1}} dt.$$

An example of when this method works well is the following: Consider the sum

$$\sum_{n\geq 0} \frac{(-1)^n n^n}{e^n n!}.$$

The exact same methods as above applies to this problem, and we have that this sum equals

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \sum_{n\geq 0} \frac{(-1)^n \exp(n(s-1))}{s^{n+1}} ds \;=\; \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{1}{s+\exp(s-1)} ds.$$

Now, for $\mathrm{Re}(s) \leq 1$, we have that

$$|s| \;\geq\; |\exp(s-1)|,$$

with equality holding only when $s = 1$. Thus, by Rouche's theorem, s and $s + \exp(s-1)$ have the same number of zeros in $\mathrm{Re}(s) \leq 1$ (actually, we are using a slightly stronger version of Rouche), counting multiplicity. Thus, $s + \exp(s-1)$ has exactly one root when $\mathrm{Re}(s) \leq 1$, and that root is of order 1 and must be real (because if α is a root, then so is $\overline{\alpha}$). Let α be that root. Then, by Cauchy's integral formula, we have that

$$\sum_{n\geq 0} \frac{(-1)^n n^n}{e^n n!} \;=\; \mathrm{Res}_{s=\alpha} \frac{1}{s+\exp(s-1)} \;=\; \frac{1}{1+\exp(\alpha-1)} \;=\; \frac{1}{1-\alpha}.$$

According to Maple, the root α of the equation

$$\alpha + \exp(\alpha - 1) \;=\; 0$$

satisfies

$$\alpha \;=\; -0.2784645428...$$

So,

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{e^n n!} \;=\; \frac{1}{1-\alpha} \;=\; 0.7821882960...$$