Practice Midterm for Math 185

1. Complex numbers, polar coordinates, and basic inequalities.

a. Prove that the largest zero of the polynomial

$$f(z) = z^5 + 3z^3 + 3z^2 + 3z + 3.$$

has norm at most 4. Hint: Use the triangle inequality as follows.

$$|f(z)| \geq \Big| |z^5| - 3|z^3 + z^2 + z + 1| \Big|.$$

b. Describe the roots of

$$f(z) = z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$$

using polar (i.e. exponential) notation.

c. Write $\mathrm{Re}\left(e^{1/z}\right)$ in terms of x and y, where $z = x + iy$.

d. If $z_1, z_2 \neq 0$, then show that

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2);$$

2. Limits.

a. Use the $\epsilon - \delta$ limit definition to prove that

$$\lim_{z \to 0} \frac{1}{z+i} = \frac{1}{i}.$$

b. Use the definition of an infinite limit to prove that

$$\lim_{z \to \infty} \frac{1}{z^2 + 1} = 0.$$

3. Derivatives (20 points).

a. Use the limit definition (not the Cauchy-Riemann equations) of the derivative to show that

$$f(z) = \overline{z}$$

is not analytic at 0.

b. Show that $f(z) = z^2 \cos(z)$ satisfies the Cauchy-Riemann equations for all z complex, where $\cos(z)$ is defined by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

c. Find the harmonic conjugate of

$$u(x, y) = 2x^3 - 6xy^2 + 5x^2 - 5y^2 + 3x,$$

and state the domain over which this conjugate exists.

d. Prove that if $f(z)$ is an analytic function in a domain D, and if $f(z)$ is real-valued for $z \in D$, then $f(z)$ must be a constant throughout D.

Bonus Question. Given a polynomial

$$f(z) = \sum_{j=0}^{k} a_j z^j,$$

(where z^0 is defined to be 1 for $z = 0$), show that

$$\int_0^1 |f(e^{2\pi i\theta})|^2 d\theta = \sum_{j=0}^{k} |a_j|^2.$$

Prove any facts you need (don't quote results from the homework).