Additive Combinatorics Homework

Ernie Croot

April 8, 2011

1. For a prime $k \geq 3$, consider the set of integers of the form

$$S \ = \ \{d_0 + d_1 k + \cdots + d_n k^n \ : \ 0 \leq d_i \leq k-2\}$$

Can S have $k-1$-term arithmetic progressions? What about k-term or $(k+1)$-term arithmetic progressions? Prove your answers.

2. Determine the size of the largest set S having the following properties: 1) $S \subseteq \{1, 2, ..., x\}$; and 2) If $x, y \in S$, then $x + y \notin S$.

3. Suppose that one has x points positioned on the unit circle, such that the minimum distance between any two of these points is at least $(100x)^{-1}$. Show that for all x sufficiently large, there exists a sector with angle width θ, containing at least $x/10^{10^{10^{100}}}$ points, such that the following holds: Let M be the number of points in this sector. Then, if we partition the sector into three equal sectors of angle width $\theta/3$, each of these sectors will contain M_1, M_2, and M_3 points, where

$$M\left(\frac{1}{3} - \frac{1}{10}\right) \ < \ M_1, M_2, M_3 \ < \ M\left(\frac{1}{3} + \frac{1}{10}\right).$$

That is, each of the three subsectors contains about the expected number of points. To solve this problem think about Roth's idea for proving sets of positive density have three-term arithmetic progressions, and think about passing to subsectors.