Hölder's Inequality and Parseval

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From the previous homework we saw that the Poisson Summation Formula fell out naturally when trying to compute the number of lattice points inside a rectangle – we didn't go looking for it... it just appeared. So, if one didn't know Poisson summation beforehand, and if all one knew were more elementary facts like "Fouirer Inversion", "Parseval", and "smoothing", one would be led inexorably to it. In a sense this means that the Poisson Summation Formula for general lattices is "visible" to the more basic approaches.

This got me to wondering what other sorts of identities and inequalities might also be "visible" to basic methods; and, more importantly, whether Fourier proofs of such results can be more easily generalized. In particular, I had wondered whether Young's inequality might "fall out naturally" somehow (like the PS Formula did). Since Young's inequality can be proved using Hölder's inequality and the tensor power trick, perhaps one should *first* try to give a new, Fourier-analytic proof of Hölder's inequality, in order to get a feel for what sorts of ideas to try. This I attempted, but have so far not proved quite what I was hoping for. Nonetheless, I think it is possible to prove at least some special cases of Hölder with Fourier methods in a way that is not just some disguised version of the usual proofs. The purpose of this note is to report an almost trivial observation along these lines:

Suppose that $f, g : \mathbb{Z}_N \to \mathbb{C}$. Then, Hölder says that

$$| < f, g > | \ \leq \ \|f\|_p \|g\|_q, \text{ where } \frac{1}{p} + \frac{1}{q} \ = \ 1.$$

Consider now the special case where $g \equiv 1$ (g is identically 1) and where $p = q = 2$, which is a special case of Cauchy-Schwarz; that is,

$$\|f\|_1 \ \leq \ N^{1/2} \|f\|_2.$$

We will prove this Fourier-analytically as follows: clearly we can suppose that $f : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ by replacing f with $|f|$. By Parseval, then, we have that

$$\|f\|_2^2 \ = \ N^{-1} \sum_{a=0}^{N-1} |\hat{f}(a)|^2 \ \geq \ N^{-1} |\hat{f}(0)|^2 \ = \ N^{-1} \|f\|_1^2.$$

The special case of Cauchy-Schwarz then follows.

What's really going on here? Basically it is that the set of characters $\chi : \mathbb{Z}_N \to \mathbb{C}^*$ form an orthogonal basis for the set of all functions from $\mathbb{Z}_N \to \mathbb{C}$. With this perspective, it seems reasonable that to prove the general Cauchy-Schwarz inequality we should work with general orthogonal bases for the vector space of all these functions (from $\mathbb{Z}_N \to \mathbb{C}$): let us begin by letting $g_0 : \mathbb{Z}_N \to \mathbb{C}$ be any function, not identically 0, such that $\|g_0\|_2 = 1$. Then, we can extend the set $\{g_0\}$ to an orthonormal basis of functions $\{g_0, g_1, ..., g_{N-1}\}$ for all functions from $\mathbb{Z}_N \to \mathbb{C}$. It follows that any $f : \mathbb{Z}_N \to \mathbb{C}$ may be written as

$$f \ = < f, g_0 > g_0 + < f, g_1 > g_1 + \cdots + < f, g_{N-1} > g_{N-1}.$$

And then,

$$\sum_{x=0}^{N-1} |f(x)|^2 \ = < f, f > = \ \sum_{i=0}^{N-1} | < f, g_i > |^2.$$

From this it follows that

$$\|f\|_2^2 \ \geq \ | < f, g_0 > |^2,$$

which is really the same as the general Cauchy-Schwarz inequality

$$| < f, g > |^2 \ \leq \ \|f\|_2^2 \|g\|_2^2$$

upon dividing g by an appropriate scaling factor to produce g_0 with $\|g_0\|_2 = 1$.

Now suppose that $p \geq 2$ is an integer, and, as before, $g \equiv 1$. Hölder then implies that

$$\|f\|_1 \ \leq \ N^{(p-1)/p} \|f\|_p;$$

or, equivalently,

$$\|f^p\|_1 \ \geq \ N^{-(p-1)} \|f\|_1^p \ = \ N^{-(p-1)} \hat{f}(0)^p,$$

where, as before, we assume $f : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$. In this situtation we can't apply Parseval; nonetheless, we have that

$$\widehat{f^p}(a) \;=\; N^{-(p-1)} \hat{f} * \hat{f} * \cdots * \hat{f}(a).$$

If we knew that $\hat{f} : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ as well, then it would follow that

$$\|f^p\|_1 \;=\; \widehat{f^p}(0) \;\geq\; N^{-(p-1)} |\hat{f}(0)|^p,$$

and we'd be done. Unfortunately, in general we don't know this.

Question. Is there some way that we can use this approach and deduce Hölder's inequality in the case $g \equiv 1$?