11 The Littlewood–Offord problem

11.1 Early results

In 1943 Littlewood and Offord published the following result. Let $z_1, \ldots, z_n$ be complex numbers such that $|z_i| \geqslant 1$ for each i; then the number of sums $\sum_{i=1}^n \varepsilon_i z_i$ $(\varepsilon_i = \pm 1)$ lying inside any circle of radius r cannot exceed

$$\frac{Cr^{2^n} \log n}{n^{1/2}} \tag{11.1}$$

for some constant C. Two years later Erdös showed, by a very nice application of Sperner's theorem, that the log term can be omitted, thus giving the best order of magnitude possible for the bound. In this chapter we shall explore variations on this problem, showing that some generalizations of Sperner's theorem are central to the development. After giving the basic ideas and the early results in this section, we shall discuss M-part Sperner theorems in Section 11.2 and then show their relevance to the Littlewood–Offord problem in Section 11.3.

There are, in fact, two equivalent formulations of the result of Littlewood and Offord described above. If $\sum_{i=1}^n z_i$ is added to each sum, the differences between sums remain unchanged, and the coefficients of the z_i are now $\delta_i = 0$ or 2. If we scale everything by a factor of $\frac{1}{2}$, we obtain the following: the number of sums $\sum_{i=1}^n \delta_i z_i$ $(\delta_i = 0$ or $1)$ lying inside any circle of *diameter* r cannot exceed the bound (11.1). This alternative formulation with coefficients 0 or 1 instead of ± 1 is sometimes more convenient.

We start our survey by considering combinations of real numbers, noting that the one-dimensional analogue of a disc of radius 1 is an interval of length 2.

Theorem 11.1.1 (Erdös 1945) Let $x_1, \ldots, x_n$ be real numbers such that $|x_i| \geqslant 1$ for each i, and let J be any interval of length 2 open at at least one end. Then the number of sums $\sum_{i=1}^n \varepsilon_i x_i$ $(\varepsilon_i = \pm 1)$ lying in J is at most $\binom{n}{[n/2]}$.

Proof Without loss of generality we can assume that all the x_i are positive; for any negative x_i can be replaced by $-x_i$. We now associate to each sum $\sum_{i=1}^{n} \varepsilon_i x_i$ the set $A = \{i : \varepsilon_i = 1\}$. If A_1 and A_2 are two such sets, and $A_1 \subset A_2$, then the corresponding sums would differ by at least 2 and so they could not both be in J. It follows that the sets A corresponding to sums in J must form an antichain; the result is therefore immediate from Sperner's theorem. □

Given any interval of length $2r$, open at at least one end, we can split it up into r intervals of length 2. Applying the above theorem to each of these intervals we obtain the following corollary.

Corollary 11.1.2 If the x_i are as in Theorem 11.1.1 and J is any interval of length $2r$ open at at least one end, then the number of sums $\sum_{i=1}^{n} \varepsilon_i x_i (\varepsilon_i = \pm 1)$ lying in J is at most $r\binom{n}{[n/2]}$. □

This bound is not, however, the best possible, for we can replace it by the sum of the r largest binomial coefficients $\binom{n}{i}$.

Theorem 11.1.3 (Erdös 1945) Let $x_1, \ldots, x_n$ be real numbers, $|x_i| \geq 1$ for each i, and let J be any interval of length $2r$, open at at least one end. Then the number of sums $\sum_{i=1}^{n} \varepsilon_i x_i$ $(\varepsilon_i = \pm 1)$ lying in J is at most

$$\sum_{i=1}^{r} \binom{n}{[(n+i)/2]}.$$

Proof As in the proof of Theorem 11.1.1, assume that each x_i is positive, and associate to each sum $\sum_{i=1}^{n} \varepsilon_i x_i$ the set $A = \{i : \varepsilon_i = 1\}$. If A_1 and A_2 are two such sets and $A_1 \subset A_2$ and $|A_2 - A_1| \geq r$, then the corresponding sums will differ by at least $2r$, so that at most one of them can be in J. The result now follows from Exercise 2.9. □

We now return to the case of complex numbers. We shall use j rather than i as a suffix so as to avoid confusion with i where $i^2 = -1$!

Theorem 11.1.4 (Erdös 1945) Let $z_1, \ldots, z_n$ be complex numbers with $|z_j| \geq 1$ for each j. Then the number of sums $\sum_{j=1}^{n} \varepsilon_j z_j$ $(\varepsilon_j = \pm 1)$