Progressions of Length 3 following Szemerédi

The purpose of the following is to give a short and easily comprehensible demonstration of the following result.

Theorem 1 (Heath-Brown, Szemerédi) *There is an absolute positive constant c such that, if N is a sufficently large integer, any set $A \subseteq [1, \dots, N]$ with density $\delta \geq (\log N)^{-c}$ contains a (non-trivial) progression of length 3.*

We make no attempt to optimise the value of c. The question of the best value that can be obtained is an interesting one, since superficial changes in the argument have an effect on c (and not just on some multiplicative constant as one might expect). Certainly one could not better $c = \frac{1}{3}$ without some serious improvements, but it is curiously difficult to pin down any reasonable value at all. Bourgain [2] has shown that $c = \frac{1}{2} - \epsilon$ is admissible, but his argument is different and quite elaborate. The result of Heath-Brown and Szemerédi is interesting because it is useful (for example in [1]) and is not too difficult.

We shall assume that the reader is familiar with the proof of Roth's result [5] on arithmetic progressions of length 3 as described in [3], and in particular we shall use the notation of that paper without further comment. Roughly speaking Roth's argument proceeds as follows. One begins by regarding A as a subset of $\mathbb{Z}_N$, a group on which one can do Fourier analysis. Write the numbers $|\hat{A}(r)|/|A|$ ($r \neq 0$) in order of size as $\alpha_1, \dots, \alpha_{N-1}$. The heart of the proof then consists in showing that either the original set A had an AP of length 3, or else A is *non-random* in the sense that $\alpha_1 \geq c\delta$ for some absolute constant c. In turn this information allows us to find a subprogression of length $\sim N^{1/2}$ on which A has increased density $\delta + c'\delta^2$. Iteration of this argument gives Roth's bound $\delta \ll (\log \log N)^{-1}$.

Our interpretation of Szemerédi's argument starts with the observation that the non-randomness obtained in Roth's argument is rather weak. We shall begin by showing that, if A has no 3-term AP, then the Fourier coefficients satisfy the stronger condition

$$\sum_{i=1}^{N-1} \alpha_i^3 \;\geq\; c.$$

We shall say that our set A is c-biased. This really is stronger than the non-randomness used in Roth's argument, as the reader may care to show using Parseval's identity. We shall use the c-biasedness of A to derive a substantial density increment by taking into consideration *several* large Fourier coefficients α_i ($i = 1, \dots, q$).

Embedding in $\mathbb{Z}_N$. As with many problems of this kind the very first step we take is to embed the problem at hand in a group on which we can do Fourier analysis. If $A \subseteq \{1, \dots, N\}$ then we may regard A as a subset of $\mathbb{Z}_N$ in a natural way, but in so doing we lose information about arithmetic progressions in A. Szemerédi proceeds by showing that, if $A \subseteq \{1, \dots, N\}$ contains no 3-term AP, there is some $N_0 \approx N$ for which A contains far fewer than the expected number of progressions when regarded as a subset of $\mathbb{Z}_{N_0}$. (By "expected number of progressions" we mean the number of progressions in a random set with the same density - recall that we are trying to show that A is highly non-random.)

Taking inspiration from [3], we offer a novel approach which allows us to show that A is in fact 2^{-27}-biased when considered as a subset of $\mathbb{Z}_N$.

Proposition 2 *Let $A \subseteq \{1, \dots, N\}$ have density δ and no nontrivial progressions of length 3. Then one of the following holds.*

(i) $N \leq 2000\delta^{-2}$*;*

(ii) A *has density at least* $16\delta/15$ *on a progression of length at least* $5N/12$*;*

(iii) A *is* 2^{-27}*-biased.*

Proof We write S for the sum $\sum_{r \neq 0} |\hat{A}(r)|^3$. Let I be the characteristic function of an interval of length $N/6$, and set

$$f(x) \;=\; \tfrac{6}{N} I * I(x - \tfrac{N}{2}).$$

Now if $\left|A \cap [\frac{5N}{12}, \frac{7N}{12}]\right| < \delta N/9$ then one of $\left|A \cap [0, \frac{5N}{12}]\right|$ and $\left|A \cap [\frac{7N}{12}, N]\right|$ is at least $4\delta N/9$ and we have Case (ii) of the proposition. Suppose then that $\left|A \cap [\frac{5N}{12}, \frac{7N}{12}]\right| \geq \delta N/9$. Let $B(x) = C(x) = A(x)f(x)$. Then one has

$$\sum_{x+z=2y} A(x)B(y)C(z) \;\leq\; |A|,$$

because any non-zero contribution to the sum comes from an arithmetic progression (x, y, z) with $y, z \in A \cap [\frac{N}{3}, \frac{2N}{3}]$. Such a progression is clearly a genuine ($\mathbb{Z}$-) progression in A and so must have $x = y = z$.

Taking Fourier transforms yields

$$\sum_r \hat{A}(r)\hat{B}(r)\hat{C}(-2r) \;\leq\; \delta N^2.$$

The contribution to the $r = 0$ term here is rather large: we have $\hat{B}(0) \geq \frac{1}{2}\left|A \cap [\frac{5N}{12}, \frac{7N}{12}]\right|$ and similarly for $\hat{C}(0)$, whence

$$\hat{A}(0)\hat{B}(0)\hat{C}(0) \;\geq\; \frac{\delta^3 N^3}{324}.$$

It follows immediately that either $N \leq 2000\delta^{-2}$, which is simply Case (i) of the proposition, or

$$\sum_{r \neq 0} |\hat{A}(r)||\hat{B}(r)||\hat{C}(-2r)| \;\geq\; \frac{\delta^3 N^3}{400}.$$

In the latter case an application of Hölder's inequality gives

$$S^{1/3}\|\hat{B}\|_3\|\hat{C}\|_3 \;\geq\; \frac{\delta^3 N^{7/3}}{400}. \tag{1}$$

Now Young's inequality tells us that

$$\|\hat{B}\|_3 \;=\; \frac{1}{N}\|\hat{A} * \hat{f}\|_3 \;\leq\; \|\hat{A}\|_3\|\hat{f}\|_1.$$

However, using Parseval's Identity,

$$\|\hat{f}\|_1 \;=\; \frac{1}{N}\sum_r |\hat{f}(r)| \;=\; \frac{6}{N^2}\sum_r |\hat{I}(r)|^2 \;=\; \frac{6}{N}\sum_x I(x)^2 \;=\; 1.$$

Hence $\|\hat{B}\|_3 \leq \|\hat{A}\|_3$, $\|\hat{C}\|_3 \leq \|\hat{A}\|_3$ and so (1) gives

$$S^{1/3}(S + |A|^3)^{2/3} \;\geq\; \frac{\delta^3 N^3}{400}.$$

The proposition follows. $\square$

We now arrive at the heart of the proof, where we show that a c-biased set has increased density on a subprogression of large size. The argument is a simplification of Szemerédi's original. We say that a progression P in $\mathbb{Z}_N$ is *non-overlapping* if its length L and common difference d satisfy $dL < N$, the point of such a definition being that a non-overlapping progression is a union of two genuine progressions.

Lemma 3 *Let $1 \leq q < N$. Then there is a nonoverlapping progression of length $\frac{1}{8}N^{1/(q+1)}$ on which A has density at least*

$$\left(1 + \tfrac{1}{4}\sum_{i=1}^{q} \alpha_i^2\right)\delta.$$

Proof Write $\alpha_i = |\hat{A}(r_i)|/|A|$. By a standard argument of Dirichlet (his "principle of the pigeons") it is possible to pick a $d \in \mathbb{Z}_N$ such that $|d|$ and all of the $|dr_i|$ are at most $N^{q/(q+1)}$. Let B be a progression of length $\frac{1}{8}N^{1/(q+1)}$ and common difference d. Our aim is to show that A has increased density on a translate of B. We begin with an observation concerning the Fourier coefficients of B. Indeed we have

$$\begin{aligned} |\hat{B}(r_i)| &= \left|\sum_x B(x)\omega^{r_i x}\right| \\ &= \left||B| - \sum_{|x|\leq |B|/2} \left(1 - \omega^{r_i x d}\right)\right| \\ &\geq |B|\left(1 - \pi|B|N^{-1/(q+1)}\right) \\ &\geq \tfrac{1}{2}|B|. \qquad (2) \end{aligned}$$

Now we can use this to deduce

$$\begin{aligned}
\max_x |A * B(x)| \cdot |A||B| &= \max_x |A * B(x)| \sum_x A * B(x) \\
&\geq \sum_x A * B(x)^2 \\
&= \frac{1}{N} \sum_r |\hat{A}(r)|^2 |\hat{B}(r)|^2 \\
&\geq \frac{1}{N} |A|^2 |B|^2 \left(1 + \tfrac{1}{4} \sum_{i=1}^{q} \alpha_i^2 \right),
\end{aligned}$$

from which it follows immediately that

$$A * B(x) \;\geq\; \delta \left(1 + \tfrac{1}{4} \sum_{i=1}^{q} \alpha_i^2 \right) |B|$$

for some x. This concludes the proof of the Lemma. $\square$

Lemma 4 *Suppose that A is c-biased, that $\delta \geq (\log N)^{-1/9}$ and that $N \geq N_0(c)$ is sufficiently large. Then there is $q \leq (\log N)^{1/2}$ such that*

$$\sum_{i=1}^{q} \alpha_i^2 \;\geq\; \tfrac{1}{9} c^{2/3} q^{1/5}.$$

Proof From the definition of c-biasedness we have that $\sum_{i=1}^{N-1} \alpha_i^3 \geq c$. Parseval's identity gives another important inequality satisfied by the α_i - namely we have $\sum_{i=1}^{N-1} \alpha_i^2 \leq \delta^{-1}$. This gives, for any m,

$$m\alpha_m^2 \;\leq\; \sum_{i=1}^{m} \alpha_i^2 \;\leq\; \delta^{-1},$$

so that

$$\sum_{i=m+1}^{N-1} \alpha_i^3 \;\leq\; \alpha_m \sum_i \alpha_i^2 \;\leq\; \delta^{-3/2} m^{-1/2}. \tag{3}$$

If $m = (\log n)^{1/2}$ and $N > N_0(c)$ then this is less than $c/2$, so that

$$\sum_{i=1}^{m} \alpha_i^3 \;\geq\; c/2.$$

Now suppose that we had $\alpha_i \leq \frac{1}{3} c^{1/3} i^{-2/5}$ for all $1 \leq i \leq m$. Then we should have

$$\begin{aligned}
\sum_{i=1}^{m} \alpha_i^3 &\leq \frac{c}{27} \sum_{i=1}^{\infty} i^{-6/5} \\
&< c/2,
\end{aligned}$$

a contradiction. It follows that there is q, $1 \leq q \leq m$, for which $\alpha_q \leq \frac{1}{3}c^{1/3}q^{-2/5}$. For this q we have

$$\begin{aligned}\sum_{i=1}^{q} \alpha_i^2 &\geq q\alpha_q^2 \\ &\geq \tfrac{1}{9}c^{2/3}q^{1/5},\end{aligned}$$

which concludes the proof. □

Using Lemmas 3 and 4 together we can find a $q \leq (\log N)^{1/2}$ and a nonoverlapping progression B of length $N' = \frac{1}{8}N^{1/(q+1)}$ such that A has density at least

$$\delta' = \left(1 + \tfrac{1}{36}c^{2/3}q^{1/5}\right)\delta$$

on B. We must now pass from nonoverlapping progressions to genuine ones.

Lemma 5 *Suppose that B is a nonoverlapping progression on which A has density $(1+\eta)\delta$. Then there is a genuine progression P of length at least $\eta\delta|B|/2$ on which A has density at least $(1+\frac{1}{2}\eta)\delta$.*

Proof Write $B = P_1 \cup P_2$ as the union of two genuine progressions, and suppose without loss of generality that $|P_1| \leq |P_2|$. If $|P_1| \leq \eta\delta|B|/2$ then we have

$$\begin{aligned}|A \cap P_2| &\geq (1+\eta)\delta|B| - |P_1| \\ &\geq (1+\tfrac{1}{2}\eta)\delta|B| \\ &\geq (1+\tfrac{1}{2}\eta)\delta|P_2|,\end{aligned}$$

and we are happy. If this is not the case then both P_1 and P_2 have length at least $\eta\delta|B|/2$, and A must have density at least $(1+\frac{1}{2}\eta)\delta$ on one of them. □

It is now an easy matter to derive the following proposition.

Proposition 6 *Suppose that A is c-biased, that $\delta > (\log N)^{-1/9}$, and that $N > N_1(c)$ is sufficiently large. Then there is a $q \leq (\log N)^{1/2}$ and a genuine progression P of length at least $N^{1/3q}$ on which A has density at least $(1+\frac{1}{72}c^{2/3}q^{1/5})\delta$.*

Using this in combination with Proposition 2 yields

Proposition 7 *Let A be a subset of $\{1,\ldots,N\}$ containing no nontrivial 3-term AP and having density δ. Then one of the following possibilities must arise:*

(i) $N \leq 2000\delta^{-2}$;

(ii) *A has density at least $16\delta/15$ on a progression of length at least $5N/12$;*

(iii) *N is less than some absolute constant ($N_1(2^{-27})$, in the above notation);*

(iv) $\delta \leq (\log N)^{-1/9}$;

(v) *There is a $q \leq (\log N)^{1/2}$ and a genuine progression P of length at least $N^{1/3q}$ on which A has density at least $(1+2^{-25}q^{1/5})\delta$.*

We are now in a position to prove Theorem 1. Let $A \subseteq \{1, \ldots, N\}$ be a set with density $\delta \geq C(\log N)^{2^{-34}}$, where C is a very large constant. We shall prove by induction on N that A contains a progression of length 3, the case of small N being vacuous. If $\delta > 1$ then the result is vacuous, so suppose $\delta \leq 1$. It is then easy to see that, in Proposition 7, only possibilities (ii) and (v) can hold. If (ii) holds then the result very obviously holds by induction. Suppose then that (v) holds, and write $N' = N^{1/3q}$, $\delta' = (1 + 2^{-25}q^{1/5})\delta$. Since $q \leq (\log N)^{1/2}$ we have $N > 3$ and so we have certainly passed to a meaningful subprogression (with room to spare). It suffices, by our inductive hypothesis, to check that

$$\delta'(\log N')^{2^{-34}} \;\geq\; \delta(\log N)^{2^{-34}}. \tag{4}$$

To see that this is so, begin by observing that one has the inequalities

$$4x^{1/5} \;\geq\; \log(3x)$$

for all $x \geq 1$ and $1 + x \geq e^{x/100}$ for $x \leq 100$. Hence if $q \leq 2^{160}$ we have

$$\log(1 + 2^{-25}q^{1/5}) \;\geq\; 2^{-32}q^{1/5} \;\geq\; 2^{-34}\log(3q),$$

which implies (4). If $q > 2^{160}$ then we have

$$\frac{1 + 2^{-25}q^{1/5}}{(3q)^{2^{-34}}} \;>\; 2^{-26}q^{1/6} \;>\; 1,$$

which also implies (4). This concludes the proof of Theorem 1. □

Comments As we mentioned earlier it is rather difficult to find a "reasonable" value of c. In spite of this, one might speculate on how large a value of c we could hope to find using this method. An important restriction comes from the fact that one can never take more than about $\log N$ Fourier coefficients into account, since if one tries to do this there turn out to be more boxes than pigeons in our application of the pigeonhole principle. This means that one must be able to truncate the inequality $\sum_{i=1}^{N-1} \alpha_i^3 \geq c$ at some $m = O(\log N)$ to get, say, $\sum_{i=1}^{m} \alpha_i^3 \geq c/2$. We did something similar (only the value of m was different) in our proof above, obtaining the inequality (3). As one can easily check, this leads to the truncated inequality we desire only if $\delta \gg (\log N)^{-1/3}$.

In my opinion it would be of interest to find a proof that $c = \frac{1}{3} - \epsilon$ is admissible – I would imagine that such a proof would necessarily involve rather more than just fiddling with numbers in the above argument. Interestingly enough we run into problems when A has one very large Fourier coefficient, say $|\hat{A}(r)| \approx \frac{1}{4}|A|$. Applying Szemerédi's argument we increase the density by a factor of less than 1/4, and in doing so we must pass to a subprogression of size as small as $N^{1/2}$. It is clear that iteration of this situation will lead to a bound on δ much weaker than $(\log N)^{1/3}$.

It is not hard to make the observation that passing to a subprogression of size $N^{1/2}$ is extremely wasteful, because we can increase the density of A on a *modular* progression of size $\sim N$. Since a modular progression is a highly structured set, this represents a strong piece of information about A. However it is hard to see how one could make this observation fit into any sort of inductive hypothesis without travelling the path taken in [2].

It is my opinion that the use of several Fourier coefficients at each stage of Szemerédi's argument is not in itself a powerful method. I believe that the overriding reason that the method improves Roth's bound is that one does not have to pass from a modular AP to a genuine one nearly so often. Since Bourgain's new method [2] was designed so as not to have this shortcoming, and because of the fact that in that paper one is dealing with $\delta \ll (\log N)^{-1/3-\eta}$, I find it rather unlikely that any mix of the two methods would give a better bound.

References

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