# The group $A_5$ is simple

February 26, 2008

## 1 Introduction

Recall that a group $G$ is said to be simple if

$$H \triangleleft G \implies H = \{e\} \text{ or } H = G.$$

That is, $G$ has only the trivial normal subgroups $\{e\}$ and $G$ itself.

Clearly, $S_n$, $n \geq 3$, is not simple, as

$$A_n \triangleleft S_n;$$

but for $n \geq 5$ it turns out that $A_n$ *is* simple. We will prove this here only for the case $n = 5$.

To do this, we will assume knowledge of the following fact, which is an assigned homework problem:

**Fact.** $A_n$, $n \geq 3$, is generated by the 3-cycles $(a\ b\ c)$. That is to say, every $\sigma \in A_n$, $n \geq 3$, can be written as

$$\sigma = C_1 C_2 \cdots C_k,$$

where each $C_i$ is one of these 3-cycles. Note that as a 3-cycle is an even permutation (e.g. $(1\ 2\ 3) = (1\ 3)(1\ 2)$ is the product of 2 transpositions), any such product $C_1 \cdots C_k$ is automatically even.

## 2 Look at cycle structure

Suppose that

$$N \triangleleft A_5, \;\; N \neq \{e\}.$$

We will show that $N = A_5$, which will prove that $A_5$ is simple, as $N$ is arbitrary.

First, from the facts that $N \neq \{e\}$ and that $N$ is a subset of $A_5$, we conclude that it contains a non-trivial even permutation $\sigma$. Consider now the disjoint cycle decomposition for $\sigma$. Clearly, the cycle structure is one of the following three types:

- $\sigma = (a\ b\ c\ d\ e)$; or,
- $\sigma = (a\ b)(c\ d)$; or,
- $\sigma = (a\ b\ c)$.

### 2.1 What if $\sigma = (a\ b\ c\ d\ e)$?

Let us consider the first case, which is where $\sigma$ is the indicated 5-cycle. Let

$$\alpha \;=\; (a\ b)(c\ d) \;\in\; A_5.$$

Then,

$$\sigma' \;:=\; \alpha\sigma\alpha^{-1} \;=\; (a\ b)(c\ d)(a\ b\ c\ d\ e)(a\ b)(c\ d) \;=\; (a\ d\ c\ e\ b).$$

Since $N$ is assumed to be normal in $A_5$, it must contain this element $\sigma'$.[1]
But since $N$ is a subgroup of $A_5$, it must also contain the product

$$\sigma\sigma' \;=\; (a\ b\ c\ d\ e)(a\ d\ c\ e\ b) \;=\; (a\ e\ c).$$

So, $N$ contains a 3-cycle $(a\ e\ c)$, and therefore we have reduced the first bullet above to the case of where $N$ contains a 3-cycle.

[1] Because, $H \triangleleft G \iff$ for every $h \in H$ and every $g \in G$ we have that $ghg^{-1} \in H$.

## 2.2 What if $\sigma = (a\ b)(c\ d)$?

As in the case of when $\sigma$ was a 5-cycle, we let

$$\beta \ = \ (a\ b\ e),$$

and then define

$$\sigma' \ := \ \beta\sigma\beta^{-1} \ = \ (a\ b\ e)(a\ b)(c\ d)(a\ e\ b) \ = \ (b\ e)(c\ d).$$

As before, we note that $N \triangleleft A_5$ implies $\sigma' \in N$; furthermore, $\sigma\sigma' \in N$, which therefore means that $N$ contains

$$\sigma\sigma' \ = \ (a\ b)(c\ d)(b\ e)(c\ d) \ = \ (a\ b\ e).$$

Again, we see that $N$ contains a 3-cycle.

## 2.3 What if $\sigma = (a\ b\ c)$?

Our goal is to show that if $N$ contains a single 3-cycle, then it contains all other 3-cycles. Since $A_n$, $n \geq 3$, is generated by the 3-cycles, this would mean that $N$ contains all the elements of $A_5$, whence

$$N \ = \ A_5.$$

Suppose that $\sigma'$ is any other 3-cycle. Then, there must be at least one number in common in the cycles $\sigma$ and $\sigma'$ (both cycles "eat up" three numbers from the list 1,2,3,4,5, so they must contain a common number); for example, if $a$ and $c$ are common to the cycles, then $\sigma'$ might look like

$$\sigma' \ = \ (a\ c\ d) \ = \ (c\ d\ a).$$

If this is the case, then we need to show that there is a $\gamma \in A_5$ such that

$$\sigma' \ = \ \gamma\sigma\gamma^{-1}.$$

In this particular case,

$$\gamma \ = \ (a\ c)(b\ d)$$

does the trick.

This $\sigma'$ was a typical example of those cycles having two numbers in common with $\sigma$. A case where it has only one number in common would be

$$\sigma' = (a\ d\ e);$$

and, in this case,

$$\sigma' = \gamma\sigma\gamma^{-1}, \text{ with } \gamma = (b\ d)(c\ e).$$

I'll leave it to you to see how to handle the case when $\sigma$ and $\sigma'$ have three numbers in common (or, if you are clever, perhaps you can see how to handle all three cases at once – one, two or three numbers in common).