A Combinatorial Method for Developing Lucas Sequence Identities

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Abstract. Let M_n be the formal sum of all length n strings composed entirely of α's and β's which have no consecutive α's, so that $M_1 = \alpha + \beta$ and $M_2 = \alpha\beta + \beta\beta + \beta\alpha$. In this paper, we present a new combinatorial method for discovering identities for Lucas sequences based on the result that M_n satisfies the recurrence $M_n = \beta M_{n-1} + \alpha\beta M_{n-2}$ when $n \geq 3$. A few applications of this approach are presented.

1 Introduction

Denote the nth Fibonacci number by F_n, where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. A well known result states that there are F_{n+2} strings of length n consisting entirely of the characters 'α' and 'β' with no consecutive α's. We define a new sequence M_n to be the formal sum of all these length n strings, so for example $M_1 = \alpha + \beta$, $M_2 = \alpha\beta + \beta\alpha + \beta\beta$, and so on. With this new

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sequence, we can generalize the result for Fibonacci sequences, stated above, to other second-order linear recurrence sequences:

Proposition 1 *For $n \geq 3$ we have that M_n satisfies the recurrence*

$$M_n \;=\; \beta M_{n-1} + \alpha\beta M_{n-2}.$$

Before we prove this, notice that when $\alpha = \beta = 1$, we have $M_n = F_{n+2}$.

Proof. Consider the formal sum M_n. Divide this sum into two subsums, where the first subsum contains all strings which begin with β, while the second subsum contains all strings which begin with α. If we remove the leading β from all the terms in the first subsum, we are left with a formal sum of all length $(n-1)$ strings which contain no consecutive α's, which is M_{n-1}; hence, the first subsum is βM_{n-1}.

Since the leading character of all terms of the second subsum is α, and since $n \geq 3$, the second character of all these terms must be β, so that there are no two consecutive α's. If we remove this leading $\alpha\beta$ from all terms in the second subsum, we are left with the sum of all strings of α's and β's of length $n-2$ which contain no consecutive α's, which is M_{n-2}; hence, the second subsum if $\alpha\beta M_{n-2}$.

The two subsums together are therefore $M_n \;=\; \beta M_{n-1} + \alpha\beta M_{n-2}$. ■

We may use this proposition to find a slightly tidier expansion for M_n by thinking of α and β as numbers instead of just formal symbols. When we do this, we can use the commutativity properties of multiplication and addition to collect terms with the same number of α's and β's.

Proposition 2 *If α and β commute then we have, for $n \geq 1$,*

$$M_n \;=\; \sum_{i=0}^{\left[\frac{n+1}{2}\right]} \binom{n-i+1}{i} \alpha^i \beta^{n-i}.$$

Proof. Since M_n is the formal sum of all length n strings of α's and β's without consecutive α's (by Proposition 1), we will count the number of such strings which contain exactly i α's and $n-i$ β's, for each $i \geq 0$.

Suppose that there are precisely x_j β's before the jth α appears in a given string. Since there are no consecutive α's in the string, we must have

$0 \leq x_1 < x_2 < \cdots < x_i \leq n - i$, and indeed each such sequence of integers x_j gives rise to a valid string of α's and β's. Thus the number of such strings is the number of ways of selecting $x_1, ..., x_i$ as above, which is $\binom{n-i+1}{i}$. Summing up over the integers $i \geq 0$ gives the result. ■

From this proposition we deduce the following classical corollary concerning Lucas sequences.

Corollary 1 *If $L_0 = 0$, $L_1 = 1$ and $L_n = aL_{n-1} + bL_{n-2}$ when $n \geq 3$, then:*

$$L_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-i}{i} b^i a^{n-1-2i}.$$

Proof. If $a = 0$ the result is trivial, so henceforth assume that $a \neq 0$. Now letting $a = \beta$ and $b = \alpha\beta$, we find that $L_2 = \beta, L_3 = \beta\beta + \alpha\beta = (\beta + \alpha)\beta$, and $L_4 = (\beta\alpha + \alpha\beta + \beta\beta)\beta$. We may thus deduce, via a simple induction hypothesis, that $L_n = M_{n-2}\beta$, when $n \geq 3$ for M_n as defined above. Therefore, by Proposition 2 above, we obtain

$$\begin{aligned} L_n &= M_{n-2}\beta = \left(\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-i}{i} \alpha^i \beta^{n-2-i} \right) \beta \\ &= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-i}{i} b^i a^{n-1-2i}. \end{aligned}$$

■

2 Another Application

In this section we establish a set of identities for Lucas sequences by again using Proposition 1.

In Proposition 1 we gave a natural combinatorial definition for the sequence M_n that only makes sense for $n \geq 1$. We can define M_n for $n \leq 0$, without a combinatorial interpretation, simply by using the recurrence definition for M_n found in Proposition 1, that is

$$M_{-(n+1)} = -\beta^{-1}\alpha^{-1}\beta M_{-n} + \beta^{-1}\alpha^{-1}M_{-(n-1)}.$$

Then, for example, $M_0 = -\beta^{-1}\alpha^{-1}\beta M_1 + \beta^{-1}\alpha^{-1}M_2 = -\beta^{-1}\alpha^{-1}\beta(\alpha+\beta) + \beta^{-1}\alpha^{-1}(\alpha\beta + \beta\alpha + \beta\beta) = 1$ and $M_{-1} = -\beta^{-1}\alpha^{-1}\beta M_0 + \beta^{-1}\alpha^{-1}M_1 = -\beta^{-1}\alpha^{-1}\beta + \beta^{-1}\alpha^{-1}(\alpha + \beta) = \beta^{-1}$. With these definitions, we obtain the following proposition, which generalizes the well-known identity that if $n = x_1+x_2+1$, then $F_n = F_{x_1}F_{x_2}+F_{x_1+1}F_{x_2+1}$, where F_n is the nth Fibonacci number as defined in the introduction.

Proposition 3 *If α and β commute, and if $n = x_1 + x_2 + \cdots + x_k + (k-1)$ where each x_i is a positive integer, then*

$$M_n \;=\; \beta^{k-1} \sum_{0\le\delta_1,\delta_2,\ldots,\delta_{k-1}\le 1} (\alpha\beta)^{\delta_1+\cdots+\delta_{k-1}} \prod_{i=1}^{k} M_{x_i-\delta_{i-1}-\delta_i},$$

where $\delta_0 = \delta_k = 0$.

The identity noted above is the case $k = 2$, $\alpha = \beta = 1$ of Proposition 3.

Proof. At first we will not assume that α and β commute. We partition a given length n string of α's and β's with no consecutive α's as follows: Begin with a block of B_1 of the first x_1 characters, followed by a single character D_1, then a block B_2 of the next x_2 characters and a single character D_2, and continue until we finish with a block B_k of x_k characters (note that we have now accounted for $x_1 + 1 + x_2 + 1 + \cdots + x_k = n$ characters; that is, the whole string). We compute the sum M_n by splitting it up into subsums, each different subsum determined by the sequence of characters $D_1, D_2, ..., D_{k-1}$. Thus, we will let $D_j = \alpha$ and $\delta_j = 1$; and $D_j = \beta$ when $\delta_j = 0$.

If $\delta_1, ..., \delta_{k-1}$ are fixed, we consider what strings are possible in the block B_j: If $D_{j-1} = \alpha$ then the first character of B_j must be β since there can be no consecutive α's; and similarly, if $D_j = \alpha$ then the last characters of B_j must be β. Otherwise the characters in B_j are free to be any string of α and β's, so long as there are no consecutive α's. Thus the formal sum of the possible strings in B_j, given by δ_{j-1} and δ_j, must be $\beta^{\delta_{j-1}}M_{x_j-\delta_{j-1}-\delta_j}\beta^{\delta_j}$ (special care needs to be taken to verify this formula when $x_j - \delta_{j-1} - \delta_j = -1$ or 0). Note that the character D_j can be written as $\alpha^{\delta_j}\beta^{1-\delta_j}$. Thus we have

$$\begin{aligned} M_n \quad = \quad & \sum_{0\le\delta_1,\delta_2,\ldots,\delta_{k-1}\le 1} (M_{x_1-\delta_1}\beta^{\delta_1})(\alpha^{\delta_1}\beta^{1-\delta_1})(\beta^{\delta_1}M_{x_2-\delta_1-\delta_2}\beta^{\delta_2})(\alpha^{\delta_2}\beta^{1-\delta_2})\cdots \\ & \cdots(\beta^{\delta_k-2}M_{x_{k-1}-\delta_{k-2}-\delta_{k-1}}\beta^{\delta_k-1})(\alpha^{\delta_{k-1}}\beta^{1-\delta_{k-1}})(\beta^{\delta_{k-1}}M_{x_k-\delta_{k-1}}). \end{aligned}$$

Assuming that α and β commute, this sum becomes

$$\sum_{0\le\delta_1,\dots,\delta_{k-1}\le 1} M_{x_1-\delta_1}\beta(\alpha\beta)^{\delta_1}M_{x_2-\delta_1-\delta_2}\beta(\alpha\beta)^{\delta_2}\cdots\beta(\alpha\beta)^{\delta_{k-1}}M_{x_k-\delta_{k-1}}$$

and the result follows when we collect the powers of $\alpha\beta$ and β using commutativity. ■

Corollary 2 *If $L_0 = 0$, $L_1 = 1$, and $L_n = aL_{n-1} + bL_{n-2}$ when $n \ge 3$, and if $n = x_1 + x_2 + \cdots + x_k + (k + 1)$ (notice the $k + 1$ – we had $k - 1$ before) where each x_i is a positive integer, then*

$$L_n = \sum_{0\le\delta_1,\delta_2,\dots,\delta_{k-1}\le 1} b^{\delta_1+\delta_2+\cdots+\delta_{k-1}}\prod_{i=1}^{k} L_{x_i+2-\delta_{i-1}-\delta_i}.$$

Proof. As in the proof of Corollary 1, we observe that when we let $\alpha = \beta$ and $b = \alpha\beta$, we get the relation $L_j = M_{j-2}\beta$, or equivalently, $M_j = \beta^{-1}L_{j+2}$. Rewriting our partition of n as $(n-2) = x_1 + \cdots + x_k + (k-1)$, and applying Proposition 3 with each $M_j = \beta^{-1}L_{j+2}$, we obtain the corollary. ■

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