Group Actions

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1 Introduction

Given a set X we say that a group G acts on X if we can think of the elements of G as being permutations on X. So, given $g \in G$ and $x \in X$ we can consider $y = g(x)$. We say that "g sends the element x to the element y". To truly be an action we need the following two proerties to hold:

1. If $e \in G$ denotes the identity element, then for every $x \in X$ we have that $e(x) = x$; that is, e fixes all the elements of X.
2. If $g, h \in G$, then the action $(gh)(x) = g(h(x))$.

It is a little bit subtle why Property 2 might be useful for applications: What property 2 is saying is that if you could factor an element $f \in G$ has $f = gh$, then the action of f on X is the same as the action of h, followed by g.

The fact that you can factor an action like this is exactly what makes simple second-order differential equations easy to solve. Let me explain: Suppose you wanted to solve the homogeneous differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y \ = \ 0.$$

One way you can do this is to re-express the left-hand-side in terms of a "differential operator" as

$$\left(\frac{d^2}{dx^2} + 3\frac{d}{dx} - 1\right) y \ = \ 0.$$

(The expression between parenthesis is the differential operator.) Now, just like with group actions, you can factor this operator into two pieces as

$$\left(\frac{d}{dx}+\theta\right)\left(\frac{d}{dx}+\theta'\right)y \;=\; \left(\frac{d}{dx}+\theta\right)\left[\left(\frac{d}{dx}+\theta'\right)y\right] \;=\; 0, \tag{1}$$

where

$$\theta \;=\; \frac{-3+\sqrt{13}}{2},\;\; \theta' \;=\; \frac{-3-\sqrt{13}}{2}.$$

If you just let

$$\psi(x) \;=\; \left(\frac{d}{dx}+\theta'\right)y \;=\; \frac{dy}{dx}+\theta' y, \tag{2}$$

which is the inner expression in (1), then you just need to solve

$$0 \;=\; \left(\frac{d}{dx}+\theta\right)\psi(x) \;=\; \frac{d\psi}{dx}+\theta\psi.$$

Well, this is easy to solve by seperation of variables: You just get

$$\psi(x) \;=\; \kappa e^{-\theta x}.$$

So, putting this into (2) you see that you have reduced the second-order differential equation to the first order equation

$$\kappa e^{-\theta x} \;=\; \frac{dy}{dx}+\theta' y,$$

and there are simple methods (integrating factors) to solve such equations.

Caution. Differential operators are *not* group actions, although they and group actions do have some similarities.

2 Homomorphism Definition and an Example

Here we give a more abstract definition of group actions, one that is more natural from the point of view of algebra: Given a group G and a set X, we say that G acts on X if there is a homomorphism

$$\varphi \;:\; G \;\to\; S_X,$$

where S_X denotes the set of permutations on the set X.

Notice here the the second property of an action as defined in the previous section is essentially the homomorphism property of φ.

Example. Suppose that $X = \{1, 2, 3\}$, which coresponds to the vertices of a triangle; and, suppose that G is the group D_3. We know that D_3 acts on the vertices $1, 2, 3$ of some triangle; and so, if we were to let φ denote this correspondance between G and these permutations on the labels $1, 2, 3$, we will have

$$\begin{aligned} \varphi(e) &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ \varphi(R) &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ \varphi(R^2) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ \varphi(F) &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\ \varphi(FR) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \\ \varphi(FR^2) &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \end{aligned}$$

where F and R have their usual meaning.

Note. Here I thought of the action of the group on the vertices a little differently than I have in class in the past. Here, the actions work from right-to-left; so, the action FR means that you first rotate, and then perform a flip. The reason for switching the order of operation here is so that φ preserves sturcture as

$$\varphi(gh) = \varphi(g) \circ \varphi(h),$$

not

$$\varphi(gh) = \varphi(h) \circ \varphi(g).$$

3 Orbit-Stabilizer Theorem

Given an element $x \in X$, look at all the places that elements of G sends x under the action of G on X. This is called the orbit of x, and is denoted orbit(x).

For example, suppose that we have a triangle, and we consider the action of the group $H = \{e, F\}$ on that triangle. Note that H is a subgroup of D_3 consisting of just one flip through the vertical axis passing through the top point of our equilateral triangle. Cetainly, if D_3 acts on the labels of the vertices of the triangle, then if we restrict to this subgroup H, we also get an action of H on X. [1]

We have $X = \{1, 2, 3\}$, the labels. Now suppose you take $x = 3$, the label of the lower right vertex in the initial configuarion (before we apply group actions to the triangle). H either sends 3 to 2, or it fixes the label 3. So, we have that the orbit of 3 is $\{2, 3\}$. Also, the orbit of 2 is $\{2, 3\}$, and the orbit of 1 is $\{1\}$, because the flip F fixes the label of the top vertex.

Another structure associated with an element $x \in X$ is the stabilizer of x, denoted stab(x). The stabilizer is the set of all $g \in G$ that fix our element x; that is,

$$\text{stab}(x) \;=\; \{g \in G \;:\; g(x) = x\}.$$

Note. I had incorrectly stated in class that stab(x) is the same as the kernel of the map

$$\psi \;:\; G \;\rightarrow\; S_{\text{orbit}(x)}.$$

That is, you can think of G not just as acting on X, but acting on the orbit of x. The kernel of this map would be the set of all $g \in G$ that fixes the *entire* orbit of x, not just x by itself. Thus, we certainly have $\ker(\psi) < \text{stab}(x)$, but these two groups are not equal!

The well-known Orbit-Stabilizer Theorem says the following:

Theorem. Suppose that a finite group G acts on a set X. Let $x \in X$ be an arbitrary element. Then,

$$|G| \;=\; |\text{orbit}(x)| \cdot |\text{stab}(x)|.$$

[1] This is a basic fact that I have not mentioned yet in my lectures: If you have a homomorphism $\varphi : G \rightarrow G'$, then you can restrict φ to a subgroup $H < G$, to produce another homomorphism $\varphi' : H \rightarrow G'$.

The proof is to look at the action of the cosets of $H = \text{stab}(x)$ on x. Say all the cosets of H in G are $a_1H, a_2H, ..., a_tH$. Then, if $g \in a_iH$, we will have that $g = a_ih$ for some $h \in H$, and

$$g(x) \ = \ (a_ih)(x) \ = \ a_i(h(x)) \ = \ a_i(x).$$

Thus, every element of a_iH sends x to the same place, namely $a_i(x)$.

If we also had that different cosets sent x to different places, then we would have that there cosets of H are in one-to-one correspondance with the orbit of x. Let us see that this is the case: Say we had that $a_i(x) = a_j(x)$. Then, we would have that $(a_j^{-1}a_i)(x) = x$. The reason that we can shift the a_j to the other side like that is that we are thinking of elements of G as being bijections from the orbit of x to itself; and, bijections are invertible.

Now, we conclude that $a_j^{-1}a_i \in \text{stab}(x) = H$, which means that $a_i \in a_jH$. So, a_iH and a_jH are not disjoint. Therefore, we conclude that have to be the same coset; that is, $a_iH = a_jH$ (because cosets are distinct if and only if they are disjoint).

Thus, different cosets of H send x to different places, and we therefore conclude that the number of different cosets of H is the size of the orbit of x. Since the number of cosets of H is $|G|/|H|$ we conclude

$$\frac{|G|}{|H|} \ = \ |\text{orbit}(x)|,$$

which proves the orbit-stabilizer theorem on multiplying through by $|H|$.