Homework 2, Analytic Methods in Number Theory

January 30, 2007

In this homework you will apply the basic combinatorial sieve I mentioned in lectures (and in lecture notes) to a few problems. Let us restate this result:

Combinatorial Sieve. For every $K \geq 1$, there exist constants $\delta > 0$, $C_1 > 0$ and $C_2 > 0$ so that the following holds for all x sufficiently large: Suppose that for each prime $p \leq x^\delta$ we identify $w(p)$ special reside classes, where

$$0 \ \leq \ w(p) \ \leq \min(p-1, K).$$

Then, suppose we "sieve out" all those integers in $[1, x]$ that happen to lie in one of these residue classes, for at least one of these primes $p \leq x^\delta$. Let S be the set that results. Then, we have that

$$C_1 x \prod_{\substack{p \leq x^\delta \\ p \text{ prime}}} \left(1 - \frac{w(p)}{p}\right) \ \leq \ |S| \ \leq \ C_2 x \prod_{\substack{p \leq x^\delta \\ p \text{ prime}}} \left(1 - \frac{w(p)}{p}\right).$$

1. Suppose that $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree d having no fixed prime factors, which means that there is no prime p that divides $f(n)$ for all n. Then, show that there exists an integer $r \geq 1$ that depends only on d, and a constant $C > 0$ that depends on d, as well as the coefficients of f, such that the following holds for all x sufficiently large:

> There are at least $Cx/\log^d x$ integers $n \leq x$ such that $f(n)$ has at most r prime factors.

(Hint: If $f(n)$ has no prime factors $\leq x^\delta$, then the number of primes that divide $f(n)$ must be small.)

2. Take problem 1, and remove the restriction that $f(n)$ has no fixed prime factors. Explain why the parameter r must now depend on the coefficients of f, and not just on its degree; and further, show that when r is allowed to depend on the coefficients of f, we still can derive a lower bound $Cx/\log^d x$ for the number of $n \leq x$ where $f(n)$ has at most r prime factors.

***3.** Does there exist a family of polynomials, one for each degree $d \geq 1$, for which the bound of $Cx/\log^d x$ is tight? That is, for each d, does there exists a polynomial $f(x) \in \mathbb{Z}[x]$, such that for every $r \geq 1$, there are at most $C_r x/\log^d x$ integers $n \leq x$ where $f(x)$ has at most r prime factors? (Don't worry about turning in this problem, as it is a harder problem than the others.)