# Homework for Analytic Methods in Number Theory

September 30, 2008

**1.** Is the Prime Number Theorem (which asserts that $\pi(x) \sim x/\log x$) equivalent to the following estimate ? :

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} \frac{1}{p} \ = \ \log\log x + C + o\left(\frac{1}{\log x}\right),$$

where $C$ is a certain constant. If so, provide a proof.

**2.** In this problem I will walk you through the proof that

$$\tau(n) \ < \ 2^{(1+o(1))\log n/\log\log n},$$

where $\tau(n)$ is the divisor function, which gives the number of positive integer divisors of the positive integer $n$.

**Step A.** Explain why at least one of the $n \leq x$ which maximizes $\tau(n)$ must be a product of consecutive primes to powers; that is, $n = 2^{a_2}3^{a_3}\cdots p_k^{a_k}$, where $a_i \geq 1$ and $p_k$ is the $k$th prime number.

**Step B.** Building on what was shown in step A, explain why $p_k \leq (1 + o(1))\log x$.

**Step C.** Write $\tau(n) = \Delta_1\Delta_2$, where

$$\Delta_1 \ = \prod_{\substack{p < (\log x)/(\log\log x)^3 \\ p \text{ prime}}} (a_p + 1), \ \ \Delta_2 \ = \prod_{\substack{(\log x)/(\log\log x)^3 \leq p < p_k \\ p \text{ prime}}} (a_p + 1).$$

By finding bounds on $a_p$, explain why

$$\Delta_1 \ = \ 2^{o((\log x)/\log\log x)}.$$

**Step D.** Explain why

$$\Delta_2 \ \leq \ 2^{\sum_{\substack{(\log x)/(\log\log x)^3 \leq p \leq p_k \\ p \text{ prime}}} a_p}.$$

**Step E.** Show that

$$\sum_{\substack{(\log x)/(\log\log x)^3 \leq p \leq p_k \\ p \text{ prime}}} a_p \ < \ \frac{(1+o(1))\log x}{\log\log x}.$$

(Hint: What is an upper bound for this sum if the weight is $a_p \log p$, instead of $a_p$ ?) Deduce the bound

$$\tau(n) \ < \ 2^{(1+o(1))(\log n)/\log\log n}.$$

**3.** Use whatever method you like to give a "good" estimate (perhaps with an error as good as $O(x \log x)$ or perhaps $O(x)$) for the sum

$$\sum_{n \leq x} \sigma(n),$$

where $\sigma(n)$ is the sum of divisors of $n$ – i.e. $\sigma(n) = \sum_{d|n} d$. Two methods that come to mind are: The convolution method and the hyperbola method.