NOTES ON PROVING ROTH'S THEOREM USING BOGOLYUBOV'S METHOD

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This is the model Bogolyubov proof of Roth's theorem, this version using Fourier analysis. The great thing is that the Fourier analysis can be replaced by combinatorics. We shall obtain the following bound for $r_3(N)$.

Theorem 0.1. *Let N be a positive integer. Then*

$$r_3(N) \ll \frac{N}{\exp(c\sqrt{\log\log N})}.$$

The proof is surprisingly straightforward. Take a set $A \subseteq \{1, \ldots, N\}$ that is free from three-term arithmetic progressions and embed A into $\mathbb{Z}/p\mathbb{Z}$ for some prime p between $2N$ and $4N$. Let us denote the density of A in $\mathbb{Z}/p\mathbb{Z}$ by α, and define

$$R := \{\gamma : |\widehat{1_A}(\gamma)| \geqslant \delta_1 \alpha\}$$

and

$$B := \{x : |\gamma(x) - 1| \leqslant \delta_2 \text{ for all } \gamma \in R\}.$$

Let $t \in B$ and consider the quantity

$$\mathbb{E}_x |1_A * 1_A(x+t) - 1_A * 1_A(x)|^2.$$

By Parseval's identity this is equal to

$$\sum_\gamma |\widehat{1_A}(\gamma)|^4 |\gamma(t) - 1|^2,$$

which, splitting the sum up into ranges depending on whether γ lies in R or not, gives us that

$$\mathbb{E}_x |1_A * 1_A(x+t) - 1_A * 1_A(x)|^2 \leqslant 4(\delta_1^2 + \delta_2^2)\alpha^3.$$

Thus averages involving $1_A * 1_A$ are approximately translation invariant by a large Bohr set. We use this observation as follows. Let P be a symmetric arithmetic progression containing 0 in B: we can find such an arithmetic progression of length at least

$$\delta_2 p^{\alpha\delta_1^2}$$

by a pigeonholing argument. Let $Q \subseteq P$ be a symmetric sub-progression of P of the same step, containing 0, but of size $(|P|+1)/2$, and define

$$\mu_Q(x) := 1_Q(x)/\mathbb{E}1_Q$$

to be the uniform probability measure supported on Q. Note that $Q + Q \subseteq B$. Consider the function $1_A * \mu_Q(x) = \mathbb{E}_{y\in Q} 1_A(x-y)$. If this is larger than 8α for some x, then we have a density increment on a translate of Q. So suppose that $1_A * \mu_Q(x) \leqslant 8\alpha$ for all x. Define

$$f := 1_A * \mu_Q / 8\alpha,$$

so that $0 \leqslant f \leqslant 1$, $\mathbb{E}f = 1/8$ and $T_3(f) = (8\alpha)^{-3}T_3(1_A * \mu_Q)$. We claim that $T_3(1_A * \mu_Q)$ must be small, which will contradict the fact the average of f is so large. Indeed,

$$T_3(1_A * \mu_Q) = \mathbb{E}_x 1_A * \mu_Q * 1_A * \mu_Q(2x) 1_A * \mu_Q(x),$$

so

$$\begin{aligned}
&|T_3(1_A * \mu_Q) - \mathbb{E}_x 1_A * 1_A(2x) 1_A * \mu_Q(x)| \\
&\qquad \leqslant (\mathbb{E}_x |1_A * \mu_Q * 1_A * \mu_Q(2x) - 1_A * 1_A(2x)|^2)^{1/2} (\mathbb{E}_x 1_A * \mu_Q(x)^2)^{1/2} \\
&\qquad \leqslant (\mathbb{E}_{y,z\in Q}\mathbb{E}_x |1_A * 1_A(2x - y - z) - 1_A * 1_A(2x)|^2)^{1/2} (8\alpha \mathbb{E}_x 1_A * \mu_Q(x))^{1/2} \\
&\qquad \leqslant 2(\delta_1 + \delta_2)\alpha^{3/2} \cdot (8\alpha^2)^{1/2} \\
&\qquad \leqslant 6(\delta_1 + \delta_2)\alpha^{5/2}.
\end{aligned}$$

Furthermore,

$$\mathbb{E}_x 1_A * 1_A(2x) 1_A * \mu_Q(x) = \mathbb{E}_{y\in Q}\mathbb{E}_x 1_A * 1_A(2x + 2y) 1_A(x),$$

and

$$|\mathbb{E}_{y\in Q}\mathbb{E}_x 1_A * 1_A(2x + 2y) 1_A(x) - \mathbb{E}_x 1_A * 1_A(2x) 1_A(x)| \leqslant 2(\delta_1 + \delta_2)\alpha^2.$$

Thus

$$|T_3(1_A * \mu_Q) - T_3(A)| \leqslant 8(\delta_1 + \delta_2)\alpha^2,$$

and so

$$T_3(f) \leqslant (8(\delta_1 + \delta_2)\alpha^2 + \alpha/p)/(8\alpha)^3 \leqslant (\delta_1 + \delta_2)/\alpha.$$

But $\mathbb{E}f \geqslant 1/8$ and so $T_3(f)$ must be large, as follows from an averaging procedure along the lines of Varnavides' theorem and numerical evaluation of $r_3(M)$ for small-ish M. This is a contradiction if we choose $\delta_1 = \delta_2 = \alpha/C$ for some large absolute constant C. Thus we must have that there is some x for which

$$\frac{|A \cap (x - Q)|}{|Q|} = 1_A * \mu_Q(x) > 8\alpha,$$

where Q is an arithmetic progression of length

$$|Q| \geqslant c\alpha p^{\alpha^3/C}.$$

Unwrapping this progression from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}$ and then rescaling, we obtain a subset A' of $\{1, \ldots, |Q|\}$ of density at least $2\delta_0 = 2|A|/N$ that is also 3AP-free. We can iterate this procedure at most $\log_2(1/\delta_0)$ times before the density reaches 1, which would constitute a contradiction if the procedure is valid at each stage. This will be the case if N is chosen large enough, and the claimed relationship between N and $\delta_0 = r_3(N)/N$ will do.

(Note that although we got a density increment by a factor of 8 (times α) on an arithmetic progression in $\mathbb{Z}/p\mathbb{Z}$, this lead only to an increment of at least 2 times the original density of A in $\{1, \ldots, N\}$.)