The Convolution Method of Evaluating Sums of Multiplicative Functions

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1 Introduction

In this document we consider the problem of estimating

$$S_1(Q) \;=\; \sum_{\substack{q \leq Q \\ q \text{ square-free}}} \frac{q}{\varphi(q)},$$

and

$$S_2(Q) \;=\; \sum_{\substack{q \leq Q \\ q \text{ square-free}}} \frac{1}{\varphi(q)}.$$

I know of several simple methods for estimating $S_1(Q)$ to within an error of about $O(Q \log^{-A} Q)$ (for some $A > 0$); however, to get the error all the way down to $O((\log Q)\sqrt{Q})$ requires some work, and in the next section, we will give such an estimate, namely $S_1(Q) = Q + O((\log Q)\sqrt{Q})$. It may be possible to give some clever, ad hoc proof of an estimate this sharp, but the proof we present will use a general approach that I like to call the "convolution method". It is possible to use much more sophisticated methods, such as what are called "contour methods", but we will not concern ourselves with these in this course. Before we embark on our proof of our estimate for $S_1(Q)$, we show how to use such an estimate to produce one for $S_2(Q)$. To do this, we will require the following simple theorem:

Theorem 1 (Abel Summation Formula) *Suppose that $f(n)$ is supported on the positive integers and that $g(t)$ is a function supported on the reals* *having a first derivative for $t \geq 1$. For each real number x, define*

$$F(x) \;=\; \sum_{n \leq x} f(n), \quad \text{and} \;\; G(x) \;=\; \sum_{n \leq x} g(n).$$

Note here that $F(x) = G(x) = 0$ for all $x < 1$ (because the sums are empty).

We have the following identity

$$\sum_{1 \leq n \leq x} f(n)g(n) \;=\; g(x)F(x) - \int_1^x g'(t)F(t)dt.$$

Proof. One way to prove this identity is to observe that the right-hand-side is linear in $F(t)$; so, it suffices to prove that both sides hold for when $f(n)$ is zero for all $1 \leq n \leq x$, except for a single value $n \leq x$ where $f(n) = 1$. In this case, the left-hand-side will just equal $g(n)$; and, the right-hand-side will be

$$g(x) - \int_1^x g'(t)F(t)dt \;=\; g(x) - \int_n^x g'(t)dt \;=\; g(x) - (g(x) - g(n)) \;=\; g(n).$$

So, both sides match, and the theorem follows.

Another way to prove the identity, and which is more general and easier to remember, is to use Riemann-Stiljes integration and integration-by-parts: We have that

$$\begin{aligned} \sum_{1 \leq n \leq x} f(n)g(n) \;&=\; \int_{1-\epsilon}^x g(t)dF(t) \\ &=\; g(t)F(t)|_{1-\epsilon}^x - \int_{1-\epsilon}^x g'(t)F(t)dt \\ &=\; g(x)F(x) - \int_1^x g'(t)F(t)dt. \quad \blacksquare \end{aligned}$$

Now, using this theorem, we show how to recover $S_2(Q)$ from $S_1(Q)$: If we let $f(n) = n/\varphi(n)$ for n square-free and 0 if n is not square-free, and let $g(t) = 1/t$, then

$$\begin{aligned} S_2(Q) \;=\; \sum_{1 \leq q \leq Q} f(q)g(q) \;&=\; \frac{S_1(Q)}{Q} + \int_1^Q \frac{S_1(t)}{t^2}dt \\ &=\; 1 + o(1) + \int_1^Q \frac{t + O((\log t)\sqrt{t})}{t^2}dt \\ &=\; \log Q \;+\; O(1). \end{aligned}$$

2 The Convolution Trick, and Estimation of $S_1(Q)$

Given a multiplicative function, one way of evaluating sums involving this function is to use the "convolution method". Basically, we want to find two functions $u(n)$ and $v(n)$ so that for q square-free, $q/\varphi(q) = (u * v)(q)$. One such pair of functions is $v(n) = 1$ and

$$u(n) \;=\; \begin{cases} 1/\varphi(n), & \text{if } n \text{ square} - \text{free;} \\ 0, & \text{otherwise.} \end{cases}$$

These functions are obviously multiplicative; and so, to check that $q/\varphi(q) = (u * v)(q)$ for square-free q, it suffices to prove that it holds for q prime: For this case, we have

$$\frac{q}{\varphi(q)} \;=\; 1 + \frac{1}{q-1} \;=\; \sum_{d|q} u(d)v(n/d) \;=\; (u * v)(q),$$

which is just what we wanted.

Now, then, expanding $S_1(Q)$ gives

$$\begin{aligned} S_1(Q) \;&=\; \sum_{\substack{q \leq Q \\ q \text{ square-free}}} \sum_{d|q} \frac{1}{\varphi(d)} \\ &=\; \sum_{\substack{d \leq Q \\ d \text{ square-free}}} \frac{1}{\varphi(d)} \sum_{q \leq Q,\ d|q} \mu^2(q). \end{aligned} \tag{1}$$

2.1 Treatment of the Inner Sum

The inner sum in this expansion of $S_1(Q)$ also will require an application of the convolution trick: We may write

$$\mu^2(q) \;=\; \sum_{e^2|q} \mu(e). \tag{2}$$

Let us now see that this formula holds: The right hand side is a convolution $(\alpha * 1)(q)$, where $\alpha(n)$ equals 0 if n is not a square, and equals $\mu(m)$ for $n = m^2$. So, since convolutions of multiplicative functions is multiplicative, to verify that the formula holds, it suffices to prove it for q equal to a prime power. It is easy to check that for p prime, $(\alpha * 1)(p^a) = 0$ for $a \geq 2$, and equals 1 for $a = 0, 1$. So, the formula (2) holds.

Thus, the inner sum in (1) is

$$\begin{aligned} \sum_{\substack{q \leq Q \\ d|q}} \sum_{e^2|q} \mu(e) &= \sum_{e^2 \leq Q} \mu(e) \sum_{\substack{q \leq Q \\ d|q,\ e^2|q}} 1 \\ &= \sum_{e^2 \leq Q} \frac{\mu(e) Q}{[d, e^2]} + O(\sqrt{Q}), \end{aligned} \tag{3}$$

where $[d, e^2]$ is the lcm of d and e^2. The error term here comes from the fact that there are $Q/[d, e^2] + O(1)$ integers $\leq Q$ that are divisible by d and by e^2.

Now, extending this last sum to all $e^2 \geq 1$, we get that the last line of (3) is

$$Q \sum_{e^2 \geq 1} \frac{\mu(e)}{[d, e^2]} + O(\sqrt{Q}).$$

Here we got an additional error term (besides the one in (3)) when we added the terms with $e^2 > Q$.

This last sum can be expressed as an Euler product as follows:

$$\begin{aligned} Q \sum_{e^2 \geq 1} \frac{\mu(e)}{[d, e^2]} &= \frac{Q}{d} \sum_{e^2 \geq 1} \mu(e) \prod_{\substack{p|e \\ p\nmid d}} \frac{1}{e^2} \prod_{p|(d,e)} \frac{1}{p} \\ &= \frac{Q}{d} \prod_{p \nmid d} \left(1 - \frac{1}{p^2}\right) \prod_{p|d} \left(1 - \frac{1}{p}\right) \\ &= \frac{Q}{d} \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p}\right) \\ &= \frac{Q}{d\zeta(2)} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1}, \end{aligned}$$

where

$$\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

Resuming our estimation of $S_1(Q)$, we find that

$$S_1(Q) \;=\; \frac{Q}{\zeta(2)} \sum_{\substack{d \le Q \\ d \text{ square-free}}} \frac{1}{d\varphi(d)} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \;+\; O((\log Q)\sqrt{Q}).$$

Now, one can show that for a square-free d,

$$\frac{1}{d\varphi(d)} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \;=\; \frac{1}{\prod_{p|d} p^2 - 1} \;=\; O\left(\frac{1}{d^2}\right).$$

Thus, we may extend the range of summation in our expression for $S_1(Q)$ to all $d \ge 1$, and produce only a small error: We have

$$\begin{aligned}
S_1(Q) \;&=\; \frac{Q}{\zeta(2)} \sum_{\substack{d \ge 1 \\ d \text{ square-free}}} \prod_{p|d} \frac{1}{p^2 - 1} \;+\; O((\log Q)\sqrt{Q}) \\
&=\; \frac{Q}{\zeta(2)} \prod_{p \text{ prime}} \left(1 + \frac{1}{p^2 - 1}\right) \;+\; O((\log Q)\sqrt{Q}) \\
&=\; \frac{Q\zeta(2)}{\zeta(2)} \;+\; O((\log Q)\sqrt{Q}) \\
&=\; Q \;+\; O((\log Q)\sqrt{Q}).
\end{aligned}$$