Homework 4, Analytic Methods in Number Theory

February 21, 2007

1. In this problem you will use the Poisson summation formula to give an analytic proof of a certain algebraic expression called a "Gauss sum": Fix an odd integer q, and consider

$$G \ := \ \sum_{n=0}^{q-1} e^{2\pi i n^2/q}.$$

In this homework you will give a proof that

$$G \ = \ \begin{cases} \sqrt{p}, & \text{if } q \equiv 1 \pmod 4; \\ i\sqrt{p}, & \text{if } q \equiv 3 \pmod 4). \end{cases}$$

First, we recall that the Poisson summation formula says that

$$\sideset{}{'}\sum_{A \leq n \leq B} f(n) \ = \ \sum_{n=-\infty}^{\infty} \int_A^B f(t) e^{2\pi i n t} dt.$$

where Σ' indicates that the terms $n = A$ and $n = B$ each get contribution $f(A)/2$ and $f(B)/2$, respectively – all other terms are correct.

Step 1. First, explain why

$$G \ = \ \sideset{}{'}\sum_{0 \leq n \leq q} e^{2\pi i n^2/q}$$

So, here we are taking $A = 0$ and $B = q$.

Step 2. Verify the identity

$$G \;=\; q \sum_{n=-\infty}^{\infty} e^{-\pi i q n^2/2} \int_{n/2}^{n/2+1} e^{2\pi i q y^2}\, dy.$$

Step 3. In the previous step observe that $e^{-\pi i q n^2/2}$ is 1 if n is even, and is i^{-q} if n is odd. So, if you divide the series into n even and odd, you can show that

$$G \;=\; q(1+i^{-q}) \int_{-\infty}^{\infty} e^{2\pi i q y^2}\, dy.$$

Show that this is the case. There is an issue here about the convergence of this integral – show that as long as $Y, Z \to \infty$,

$$\int_{-Y}^{Z} e^{2\pi i q y^2}\, dy \;\to\; \int_{-\infty}^{\infty} e^{2\pi i q y^2}\, dy.$$

Step 4. Upon setting $y = q^{-1/2}u$, show that

$$G \;=\; (1+i^{-q}) C q^{1/2},$$

where C is a certain integral. Then, plugging in $q = 1$, show that $C = (1+i^{-1})^{-1}$, and deduce

$$G \;=\; \frac{1+i^{-q}}{1+i^{-1}}\sqrt{q} \;=\; \begin{cases} \sqrt{q}, & \text{if } q \equiv 1 \pmod 4; \\ i\sqrt{q}, & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

whence our formula for G stated earlier.

2. In this problem you will use contour integration to find the exact value of

$$S \;:=\; \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}.$$

Step 1. First, realize that the residue at s equal to an integer n of $\pi\cot(\pi s)$ is 1; and, from this deduce, explain how

$$\frac{1}{2\pi i}\int_B \frac{\pi\cot(\pi s)}{s^2+1} ds$$

is related to S, where B is a certain box whose corners are given by

$$-(T+1/2)+iT, \quad -(T+1/2)-iT, \quad (T+1/2)+iT, \quad (T+1/2)-iT,$$

where T is a large positive integer. To answer this problem, keep in mind that $(1+s^2)^{-1}$ has simple poles at $s=\pm i$.

Step 2. Explain why as $T \to \infty$, the integral around the box is 0. From this observation, and from Cauchy residue formula, express S in terms of the residues at $s=\pm i$ of $\pi(1+s^2)^{-1}\cot(\pi z)$.