Homework 5, Analytic Number Theory

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In this homework you will apply a generlization of the standard Chevalley-Warning theorem to prove a result in combinatorics. (I thank Trevor Wooley for pointing out a crude version of this idea to me.)

First, let us recall what this generalized theorem says: Suppose that we have a sequence of polynomials

$$f_1(x_1, ..., x_n),\ ...,\ f_m(x_1, ..., x_n)\ \in\ \mathbb{F}_p[x_1, ..., x_n],$$

Further, suppose that these polynomials have degrees $d_1, ..., d_m$. Then, if

$$d_1 + \cdots + d_m\ <\ n,$$

we have that the number of solution vectors $\vec{x} = (x_1, ..., x_n)$ to the simultaneous congruence

$$f_1(\vec{x})\ \equiv\ f_2(\vec{x})\ \equiv\ \cdots\ \equiv\ f_m(\vec{x})\ \equiv\ 0 \pmod p$$

is divisible by p.

Thus, if $(0, ..., 0)$ is a "trivial solution" to the system of polynomial congruences, then there must exist at least one "non-trivial solution". Now we give the application.

Problem. Suppose that A is an $m \times n$ matrix with integer entries. We will show that there exists a vector $\vec{x} = (x_1, ..., x_n)$, with $x_i = 0$ or 1, such that

$$Ax\ \equiv\ \mathbf{0} \pmod p,$$

provided that

$$m\ <\ \frac{n}{p-1}.$$

There are many nice consequences of this fact. One has to do with bipartite graphs, which is mentioned in one of the steps below.

Step 1. Given that the i, j entry of A is denoted as $A_{i,j}$, show that $Ax \equiv 0 \pmod p$ has a non-trivial solutions with the $x \in \{0,1\}^n$ if and only if the following system has a non-trivial solution:

$$\begin{array}{rcl} A_{1,1}x_1^{p-1} + \cdots + A_{1,n}x_n^{p-1} & \equiv & 0 \pmod p \\ A_{2,1}x_1^{p-1} + \cdots + A_{2,n}x_n^{p-1} & \equiv & 0 \pmod p \\ & \vdots & \\ A_{m,1}x_1^{p-1} + \cdots + A_{m,n}x_n^{p-1} & \equiv & 0 \pmod p. \end{array}$$

Step 2. By defining the polynomial in the ith equation above as $P_i(x_1, ..., x_n)$, use Chevalley-Warning to prove that the equation $Ax \equiv 0$, $x \in \{0,1\}^n$, has a non-trivial solution so long as

$$m < \frac{n}{p-1},$$

as claimed.

Step 3. Use your solution to step 2 to prove the following combinatorial result: Suppose that we have a complete bipartite graph with m vertices on the left-hand-side, and n vertices on the right-hand-side. Further, suppose that each of the edges have integer weights. Prove that if

$$m < \frac{n}{p-1},$$

then there is some non-empty subset of the n vertices on the right, such that if we delete those vertices, and all the edges they are connected to, then each of the sums of weights of edges of each of the m vertices on the left, is divisible by p.