Homework 6, Analytic Methods in Number Theory

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In this problem you will prove the Brun-Titchmarsh theorem using the Large Sieve of Montgomery and Vaughan. Let us recall what this version of the large sieve says: Suppose that

$$S \ \subseteq \ \{M+1, ..., M+N\},$$

such that for each prime $p \leq Q$ we have that S *fails* to occupy

$$0 \ \leq \ \omega(p) \ \leq p$$

residue classes mod p. Then,

$$|S| \ \leq \ \frac{N + 3Q^2}{\sum_{r \ \text{square-free}}^{r \leq Q} \prod_{p|r} \frac{\omega(p)}{p - \omega(p)}}.$$

(There are better versions of the large sieve known, but this version is good enough for our purposes.) Throughout the proof we will use

$$Q \ = \ \delta\sqrt{N},$$

where $\delta > 0$ will depend on a parameter $\epsilon > 0$ stated later; so, we will be using

$$|S| \ < \ \frac{(1+\delta)N}{\sum_{r \ \text{square-free}}^{r \leq Q} \prod_{p|r} \frac{\omega(p)}{p - \omega(p)}}.$$

Later, there will be a factor $(1 + 2\delta)$ in the numerator – this new factor is not a mistake!

The Brun-Titchmarsh theorem, at least the version you will prove, says

Brun-Titchmarsh Theorem. For $\epsilon > 0$ and $k \geq k_0(\epsilon)$, if $x \geq 0$ and $q \geq 2$, then

$$\pi(x + kq; a, q) \ - \ \pi(x; a, q) \ \leq \ (2 + \epsilon)\pi(k)\frac{q}{\phi(q)}.$$

(This is slightly different from the version I stated in class. This version is easier for the purposes of a nice homework problem.)

Here, we use the notation

$$\pi(x; a, q) \ = \ |\{p \leq x,\ p \equiv a \pmod{q} \ : \ p \text{ is prime}\}|.$$

Step 1. First, identify each prime number $P \in (x, x + kq]$ satisfying

$$P \ = \ x + nq$$

with the number n. Note that this gives us a subset S of $\{1, ..., k\}$ consisting of these values of n. Now, for each prime $p \leq Q$ such that p does not divide q, explain why there is a certain residue class mod p that contains at most one element of S.

Step 2. For each prime $p \leq Q$, remove from S those elements that lie in the single special residue class you found in step 1. Call the new set S'. Note that $|S'| = |S| - O(Q/\log Q)$.

****Step 3.** You can skip this step if you are not up to it: Prove the estimate

$$\sum_{\substack{r \leq R,\ \gcd(r,q)=1 \\ r \text{ square-free}}} \frac{1}{\varphi(r)} \ \sim \ \frac{\varphi(q)}{q} \log R.$$

One way to prove this cheaply is to use the estimate

$$\sum_{\substack{r \leq R \\ r \text{ square-free}}} \frac{1}{\varphi(r)} \ \sim \ \log R,$$

and then some inclusion-exclusion.

Step 4. From steps 2 and 3, and the Large Sieve, conclude that for any fixed $\delta > 0$ (remember, $Q = \delta\sqrt{k}$) if k is sufficiently large

$$|S'| \;\leq\; \frac{(1+2\delta)k}{\frac{\varphi(q)}{q}\log k} \;=\; (1+2\delta)\frac{q}{\varphi(q)}\frac{k}{\log k}.$$

Step 5. Put together all the above steps, and conclude that the Brun-Titchmarsh theorem is true.