ON NON-INTERSECTING ARITHMETIC PROGRESSIONS

ERNEST S. CROOT III

ABSTRACT. Let $L(c,x) = e^{c\sqrt{\log x \log\log x}}$. We prove that if $a_1 \pmod{q_1}, ..., a_k \pmod{q_k}$ are a maximal collection of non-intersecting arithmetic progressions, with $2 \le q_1 < q_2 < \cdots < q_k \le x$, then

$$\frac{x}{L(\sqrt{2}+o(1),x)} \ < \ k \ < \ \frac{x}{L(1/6-o(1),x)}.$$

In the case for when the q_i's are square-free, we obtain the improved upper bound

$$k \ < \ \frac{x}{L(1/2-o(1),x)}.$$

I. INTRODUCTION

Suppose that $a_1 \pmod{q_1}, a_2 \pmod{q_2}, ..., a_k \pmod{q_k}$ is a collection of arithmetic progressions, where $2 \le q_1 < \cdots < q_k \le x$, with the property that

$$\{a_i \pmod{q_i}\} \cap \{a_j \pmod{q_j}\} \ = \ \emptyset, \text{ if } i \ne j.$$

We say that such a collection of arithmetic progressions is disjoint or non-intersecting. Let $f(x)$ be the maximum value for k, maximized over all choices of progressions $a_i \pmod{q_i}$. Define

$$L(c,x) \ := \ \exp(c\sqrt{\log x \log\log x}),$$

and define

$$\psi(x,y) \ := \ \#\{n \le y \ : \ p \text{ prime}, \ p|n \Longrightarrow p \le y\}, \text{ and}$$
$$\psi^*(x,y) \ := \ \#\{n \le y \ : \ p \text{ prime}, \ p^a|n \Longrightarrow p^a \le y\}.$$

In [3], Erdős and Szemerédi prove that

$$\frac{x}{\exp\left((\log x)^{1/2+\epsilon}\right)} \ < \ f(x) \ < \ \frac{x}{(\log x)^c},$$

for some constant $c > 0$. (This result is also mentioned in [2].) Their lower bound can be refined by using more exact estimates for $\psi(x, L(c,x))$ than was used in their paper. Specifically, as direct consequence of [Lemma 3.1, 1], we have the following estimate

Lemma 1. *For any constant $c > 0$,*

$$\psi(x, L(c,x)) = \frac{x}{L\left(1/(2c) + o(1), x\right)}. \tag{1}$$

We also have the same estimate for $\psi^*(x, L(c,x))$, since

$$\begin{aligned}\psi(x, L(c,x)) > \psi^*(x, L(c,x)) > \psi(x, L(c,x)) &- \sum_{n^2 > L(c,x)} \psi(x/n^2, L(c,x)) \\ = \psi(x, L(c,x)) - O\left(\frac{x}{L\left(c/2 + 1/(2c) + o(1), x\right)}\right)&, \end{aligned} \tag{2}$$

Now, let p be the largest prime number less than or equal $L(1/\sqrt{2}, x)$. Let $q_1, q_2, ..., q_t$ be the collection of all integers $\leq x$ which are divisible by p, and whose prime power factors are all $< p$. From (1) and (2), we have that $t = x/\left(pL(1/\sqrt{2} + o(1), x)\right) = x/L(\sqrt{2} + o(1), x)$. For each $q_i = p\ell_r^{h_r}\ell_{r-1}^{h_{r-1}} \cdots \ell_1^{h_1}$, where $p > \ell_r^{h_r} > \ell_{r-1}^{h_{r-1}} > \cdots > \ell_1^{h_1}$ are the powers of the disctint primes dividing q_i, we choose the residue class $a_i \pmod{q_i}$ using the Chinese Remainder Theorem as follows:

$$a_i \equiv \ell_r^{h_r} \pmod{p};\ a_i \equiv \ell_{j-1}^{h_{j-1}} \pmod{\ell_j^{h_j}},\ \text{for } 2 \leq j \leq r;$$
$$\text{and finally, } a_i \equiv 0 \pmod{\ell_1^{h_1}}.$$

This is exactly the construction which appears in [3] (except that their progressions were all square-free), and it is easy to see that our choice of progressions $a_i \pmod{q_i}$ are disjoint. Thus, we have that

$$f(x) > \frac{x}{L(\sqrt{2} + o(1), x)}.$$

In this paper we will prove the following results:

Theorem 1. *If $a_1 \pmod{q_1}, ..., a_k \pmod{q_k}$ are a collection of disjoint arithmetic progressions, where the q_i's are square-free and $2 \leq q_1 < \cdots < q_k \leq x$, then*

$$k < \frac{x}{L(1/2 - o(1), x)}.$$

Corollary to Theorem 1.

$$f(x) < \frac{x}{L(1/6 - o(1), x)}.$$

Thus, we will have shown that

$$\frac{x}{L(\sqrt{2} + o(1), x)} < f(x) < \frac{x}{L(1/6 - o(1), x)}.$$

To see how the Corollary follows from Theorem 1, let $b_1 \pmod{r_1}, ..., b_{f(x)} \pmod{r_{f(x)}}$ be a maximal collection of disjoint arithmetic progressions with $2 \leq r_1 < \cdots < r_{f(x)} \leq x$. Suppose, for proof by contradicition, that for some $\epsilon < 1/6$

$$f(x) > \frac{x}{L(1/6 - \epsilon, x)}. \tag{3}$$

Write each $r_i = \alpha_i\beta_i$, where β_i is square-free, $\gcd(\alpha_i, \beta_i) = 1$, and every prime dividing α_i divides to a power ≥ 2. (Note: we may have α_i or $\beta_i = 1$.) Now, at least half of α_i's must be $\leq L(1/3, x)$, for if not we would have from our assumption (3) that

$$\begin{aligned}\frac{x}{2L(1/6-\epsilon, x)} &< f(x)/2 < \#\{r_i \ : \ \alpha_i > L(1/3, x)\} \\ &< \ x \sum_{n^2 > L(1/3,x)} \frac{1}{n^2} \prod_{p \text{ prime}} \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right) \\ &\ll \frac{x}{L(1/6, x)},\end{aligned}$$

which is impossible for x large enough in terms of ϵ. Thus, we must have that there exists an $\alpha < L(1/3, x)$ for which at least $f(x)/\left(2L(1/3, x)\right)$ of the r_i's have $\alpha_i = \alpha$. Let $R(\alpha) \subseteq \{r_1, ..., r_{f(x)}\}$ be such a collection of r_i's, where

$$|R(\alpha)| > \frac{f(x)}{2L(1/3, x)} > \frac{x}{2L(1/2-\epsilon, x)},$$

where this last inequality follows from our assumption (3). Now there must exist a residue class $b \pmod{\alpha}$ for which at least $|R(\alpha)|/\alpha$ of the progressions $b_i \pmod{r_i}$ satisfy

$$r_i \in R(\alpha), \text{ and } b_i \equiv b \pmod{\alpha}. \tag{4}$$

Thus, the arithmetic progressions $b_i \pmod{r_i/\alpha}$, where r_i satisfies (4), is a collection of $\geq |R(\alpha)|/\alpha \gg x/(\alpha L(1/2-\epsilon, x))$ disjoint progressions, with distinct square-free moduli $\leq x/\alpha$. This contradicts Theorem 1 for x sufficiently large in terms of ϵ. We must conclude, therefore, that the bound in (3) is false for all $\epsilon < 1/6$ and $x > x_0(\epsilon)$, and so the Corollary to Theorem 1 follows.

II. Proof of Theorem 1

Before we prove Theorem 1, we will need the following lemma:

Lemma 2. *There are at most $x/L(c/2 + o(1), x)$ positive integers $n \leq x$ such that $\omega(n) > c\sqrt{\log x/\log\log x}$. (Recall: $\omega(n) = \sum_{p|n, \ p \text{ prime}} 1$.), where c is some positive constant.*

Proof of Lemma 2. We observe that

$$\begin{aligned}\#\{n \leq x \ : \ \omega(n) > c\sqrt{\log x/\log\log x}\} \ &< \ x \sum_{j > c\sqrt{\frac{\log x}{\log\log x}}} \frac{\left(\sum_{\substack{p^a \leq x \\ p \text{ prime}}} \frac{1}{p^a}\right)^j}{j!} \\ &= \frac{x}{(c\sqrt{\log x/\log\log x})^{\{c+o(1)\}\sqrt{\log x/\log\log x}}} \\ &= \frac{x}{L(c/2 + o(1), x)}.\end{aligned}$$

We now resume the proof of Theorem 1. Consider the collection of all the q_i's with the properties

A. $\omega(q_i) < \sqrt{\frac{\log x}{\log\log x}}$, and

B. There exists a prime $p > L(1, x)$, such that $p | q_i$,

Let $\{r_1, ..., r_{k'}\}$ be the collection of all such q_i's satisfying A and B, and where $\{b(r_1), ..., b(r_{k'})\}$ are their corresponding residue classes.

To prove our theorem, we start with the set $S_0 = \{r_1, ..., r_{k'}\}$, and construct a sequence of subsets $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$, and a sequence of primes $p_1, p_2, ...$ (and let $p_0 = 1$), such that the for each $i \geq 1$, the following three properties hold

1. Each member of S_i is divisible by the primes $p_1, ..., p_i$,
2. There exists an integer A_i, such that for each $r_j \in S_i$, we have that $b(r_j) \equiv A_i \pmod{p_1 p_2 \cdots p_i}$.
3. $|S_i| > |S_{i-1}|/(p_i \sqrt{\log x / \log\log x})$.

We continue constructing these subsets until we reach a subset S_t which has the additional property:

4. There exists a prime $p \neq p_1, ..., p_t$, $p \geq L(1, x)$ such that at least $|S_t|/\sqrt{\log x/\log\log x}$ of the elements of S_t are divisible by p.

Let us suppose for the time being that we can construct these sets $S_1, ..., S_t$. Applying Property 3 iteratively, together with Property 4, we have that the number of elements of S_t which are divisible by p (which are already divisible by $p_1 p_2 \cdots p_t$ by Property 1) is at least

$$\frac{|S_0|}{p_1 p_2 \cdots p_t (\sqrt{\log x/\log\log x})^{t+1}} \geq \frac{|S_0|}{p_1 p_2 \cdots p_t L(1/2 + o(1), x)},$$

(Note: By Property A above we have that $t \leq \sqrt{\log x/\log\log x}$ since every element of S_0 has at most $\sqrt{\log x/\log\log x}$ prime factors.) From this, together with the fact that $p > L(1, x)$, we have

$$\begin{aligned}\frac{x}{p_1 \cdots p_k L(1, x)} &\geq \#\{n \leq x \ : \ p p_1 p_2 \cdots p_t | n\} > \#\{q \in S_t \ : \ p | q\} \\ &\geq \frac{|S_0|}{p_1 p_2 \cdots p_t L(1/2 + o(1), x)}.\end{aligned}$$

It follows that

$$|S_0| < \frac{x}{L(1/2 - o(1), x)},$$

From this, together with Lemmas 1 and 2 and the fact that the elements of S_0 satisfy A and B above, we have that

$$\begin{aligned}\frac{x}{L(1/2 - o(1), x)} > |S_0| &> k - \#\{n \leq x : \omega(n) \geq \sqrt{\log x/\log\log x}\} \\ &\quad - \psi(x, L(1, x)) \\ &> k - \frac{x}{L(1/2 - o(1), x)},\end{aligned}$$

and so

$$k < \frac{x}{L(1/2 - o(1), x)},$$

which proves our theorem.

To construct our sets S_i, we apply the following iterative procedure: suppose we have constructed the sets $S_1, ..., S_i$, which satisfy 1 through 3 as above. To construct S_{i+1}, first pick any element $r \in S_i$. Now let $e_1, ..., e_j$ be all those primes dividing $r/(p_1 \cdots p_i)$ (note: $j < \sqrt{\log x/\log\log x}$). Each element $s \in S_i$, $s \neq r$, is divisible by at least one of these primes, since otherwise $\gcd(r,s) = p_1 \cdots p_i$ and so we would have $b(r) \equiv A_i \equiv b(s) \pmod{\gcd(r,s)}$, which would mean that $\{b(r) \pmod r\} \cap \{b(s) \pmod s\} \neq \emptyset$.

Now, there must be at least $|S_i|/j > |S_i|/\sqrt{\log x/\log\log x}$ of the elements of S_i which are divisible by one of these primes e_h. Let $C_i \subseteq S_i$ be the collection of all elements S_i divisible by this prime e_h. There exists at least one residue class B $(\bmod\ e_h)$ for which more than $|C_i|/e_h > |S_i|/(e_h\sqrt{\log x/\log\log x})$ of the elements $r \in C_i$ satisfy $b(r) \equiv B \pmod{e_h}$. Now let S_{i+1} be the collection of all such $r \in C_i$, set $p_{i+1} = e_h$, and let $A_{i+1} \equiv A_i \pmod{p_1 \cdots p_i}$ and $A_{i+1} \equiv B \pmod{p_{i+1}}$ by the Chinese Remainder Theorem. Then we will have that properties 1, 2, and 3 as above follow immediately for this set S_{i+1}.

If there exists a prime $p > L(1,x)$ which divides more than $|S_{i+1}|/\sqrt{\log x/\log\log x}$ of the elements of S_{i+1}, then we set $t = i+1$ and we are finished. If not, we continue constructing these sets S_j. We are guaranteed to eventually hit upon such a prime p since all our r_j's are divisible by at least one prime $p > L(1,x)$ by property B.

References

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