Meinardus's Theorem on Generating Functions

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1 Introduction

We know that the partition function $p(n)$ has the nice generating function

$$\sum_{n\geq 0} p(n)q^n \;=\; \prod_{n\geq 1}(1-q^n)^{-1}$$

(by convention $p(0)=1$), and we discussed in class that $p(n)$ has the asymptotic formula

$$p(n) \;\sim\; \frac{1}{4n\sqrt{3}}\exp(\pi\sqrt{2n/3}).$$

There are several known proofs of this fact, but perhaps the most general theorem of this type is due to Meinardus, who produced asymptotic formulae for functions $r(n)$ defined by generating functions of the form

$$\sum_{n\geq 0} r(n)q^n \;=\; \prod_{n\geq 1}(1-q^n)^{-a_n},$$

where $a_1, a_2, ...$ is a sequence of non-negative reals (usually non-negative integers) having certain nice properties (such as that if you build a zeta function $\sum_{n\geq 1} a_n/n^s$ out of them, then that series is meromorphic in a half-plane $\mathrm{Re}(s) > -C_0$, for some $C_0 > 0$).

In his book *The Theory of Parititions*, George Andrews does a nice job stating and presenting the proof of Meinardus's theorem (except for a few notational problems) in just a few pages, but here I just work out his proof in the special case where $r(n) = p(n)$.[1] In the course of the proof I pull out the key tools used and present them in seperate subsections for easy reference (the tools, after all, are the important things to take away from the proof – not the proof itself!); however, we will not bother to give the sharpest error estimates possible using Meinardus's approach.

2 The saddle point method

Let us define

$$\Delta(q) \ := \ \sum_{k\geq 0} p(n)q^k \ = \ \prod_{k=1}^{\infty}(1-q^k)^{-1}.$$

It is clear that in order for $\Delta(q)$ to converge we must have $|q| < 1$; further, upon taking logs, and playing with series, you can show that it does indeed converge for all such q in the interior of the unit circle, and that it is analytic there. Two consequences of the fact that Δ converges in the unit circle are that $p(n) = e^{o(n)}$ and that

$$\sum_{k\geq 0} p(k)q^k \ \text{ converges absolutely in the unit circle.}$$

Using the standard fact that for an integer m,

$$\int_{-1/2}^{1/2} e^{2\pi imt}dt \ = \ \begin{cases} 1, & \text{if } m = 0; \\ 0, & \text{if } m \neq 0, \end{cases}$$

we can easily write $p(n)$ in terms of a certain integral as follows: For every $\sigma > 0$, we have

$$p(n) \ = \ \int_{-1/2}^{1/2} \Delta(e^{-\sigma-2\pi it})e^{n(\sigma+2\pi it)}dt.$$

To see that this holds, we expand this function Δ into a series, and swap the integral and sum (we can perform this swap because for $\sigma > 0$ the Δ written

[1] I think it is very likely that this is how Meinardus proved his theorem in the first place: He first proved it for $p(n)$, and then realized that he could get much more from his method.

as a series converges absolutely) to produce

$$\begin{aligned}
\sum_{k\geq 0} p(k) \int_{-1/2}^{1/2} & e^{-k(\sigma+2\pi it)} e^{n(\sigma+2\pi it)} dt \\
&= \sum_{k\geq 0} p(k) e^{\sigma(n-k)} \int_{-1/2}^{1/2} e^{2\pi it(n-k)} dt \\
&= p(n).
\end{aligned}$$

But now we are faced with a choice: Which $\sigma > 0$ should we pick to make the integral easy to estimate? Well, in a lot of problems of this type, the "main contribution" to the value of this integral comes from those t near 0, and then the contribution for t away from 0 is thrown into an "error term"; that is,

$$p(n) \ = \ \int_{-\delta}^{\delta} \Delta(e^{-\sigma-2\pi it}) e^{n(\sigma+2\pi it)} dt \ + \ E, \tag{1}$$

where

$$E \ = \ \int_{\delta<|t|\leq 1/2} \Delta(e^{-\sigma-2\pi it}) e^{n(\sigma+2\pi it)} dt.$$

Now, for some values of σ we may need to use a value of δ which is near 1/2, which would be bad; and, for some σ, we maybe could use δ close to 0, which would be fantastic. The so-called "saddle point method" is a method for locating and using a good value for σ (i.e. one which only needs small values of δ).

2.1 Why use the word "saddle" ?

Let me now explain where the words "saddlepoint method" come from: Let us say that for each $\sigma > 0$ you plot a vertical line passing from

$$(\sigma, -\delta) \ \rightarrow \ (\sigma, \delta),$$

where $\delta = \delta(\sigma)$ is the value of δ you need to guarantee that the error E much smaller than $p(n)$, say

$$|E| \ < \ p(n)/2. \tag{2}$$

Then, the shape you typically get for these sorts of problems is a "saddle" – i.e. there is one particualr value of $\sigma > 0$ which gives you the smallest value of δ, and then if you are either to the right or left of that special value the vertical lines are longer.[2] Let this special value of σ be denoted by σ_0; that is, σ_0 minimizes the value of δ so that (2) holds.

2.2 How do you find the "saddlepoint"?

What is the value of σ_0 ? Well, typically, it is the value which minimizes your integrand at $t = 0$ in your functional expansion; in our case, the "function expansion" is (1). Thus, to find a "good" value for σ, which is not always the "best" value (which would be $\sigma = \sigma_0$), we must minimize

$$\Delta(e^{-\sigma})e^{n\sigma}.$$

At the moment we have no hope of doing this, because we don't yet know how Δ grows as a function of q. We will remedy this in the next several sections by proving a very sharp formula for the function $f(\tau)$, which is just a more convenient form of the function Δ to work with, and is written as follows:

$$f(\tau) \ = \ \prod_{n\geq 1}(1-e^{-n\tau})^{-1}, \text{ where } \tau = \sigma + 2\pi i t.$$

So, expressing our above integral formulas in terms of f we find that

$$p(n) \ = \ e^{n\sigma}\int_{-1/2}^{1/2} f(\sigma+2\pi i t)e^{2\pi i n t}dt,$$

and our "very sharp formula" is given by the following theorem.

Theorem 1 *For* $0 < \sigma < 1/6$ *we have that*

- *For* $|\mathrm{Arg}(\tau)| \leq \pi/4$,

$$f(\tau) \ = \ \exp\left(\tau^{-1}\zeta(2) - \log(2\pi)/2 + \log(\tau)/2 + O(\sigma^{1/2})\right).$$

[2] Well, there are other ways to see a "saddle" in this method, but I prefer the explanation I just gave!

- *For* $\pi/4 < |\mathrm{Arg}(\tau)| \leq \pi/2$,

$$f(\tau) \ \leq \ f(\sigma)\exp\Big(-(1-1/\sqrt{2}+o(1))\sigma^{-1}\Big).$$

From this theorem, and our earlier comments, we see that we must minimize

$$e^{n\sigma}\exp(\sigma^{-1}\zeta(2)+\log(\sigma)/2).$$

So, we must minimize

$$n\sigma+\zeta(2)\sigma^{-1}+\log(\sigma)/2.$$

Taking derivatives and setting to 0, and multiplying through by σ^2 to clear denominators, we must solve

$$n\sigma^2+\frac{\sigma}{2}-\zeta(2) \ = \ 0.$$

So, keeping only the positive root, we find

$$\sigma \ = \ \frac{-1/2+\sqrt{1/4+4n\zeta(2)}}{2n} \ \sim \ \sqrt{\frac{\zeta(2)}{n}}.$$

There is no need to keep the lower order terms, because for large n we will get good estimates for $p(n)$ if we just use

$$\sigma \ := \ \sqrt{\frac{\zeta(2)}{n}}.$$

We now have all the ingredients assembled to prove the following theorem.

Theorem 2 *We have that*

$$p(n) \ = \ \frac{1+O(n^{-1/10})}{4n\sqrt{3}}\exp(\pi\sqrt{2n/3}).$$

3 Proof of Theorem 2

For some reasons that are not immediately obvious we truncate our integral up to height $\delta = \sigma^{7/5}$, and so we write

$$p(n) \;=\; e^{n\sigma} \int_{-\sigma^{7/5}}^{\sigma^{7/5}} f(\sigma + 2\pi it)e^{2\pi int}dt \;+\; E,$$

where the "error" E is just

$$E \;=\; e^{n\sigma} \left(\int_{-1/2}^{-\sigma^{7/5}} + \int_{\sigma^{7/5}}^{1/2} f(\sigma + 2\pi it)e^{2\pi int}dt \right).$$

3.1 The error E

To deal with the error E we will need to use both parts of Theorem 1.

3.1.1 The hard case: $t > \sigma^{7/5}$ and $|\mathrm{Arg}(\tau)| \leq \pi/4$

First, we consider the "hard case", which is where

$$t > \sigma^{7/5},\ \text{but}\ |\mathrm{Arg}(\tau)| \;\leq\; \pi/4.$$

In order to apply Theorem 1 we must deal with the terms $\tau^{-1}\zeta(2)$ and $\log(\tau)/2$ (it will turn out that this log term can be ignored, as it won't much affect the bounds we develop): If

$$\sigma^{7/5} \;<\; |t| \;<\; \sigma/10,\ |\mathrm{Arg}(\tau)| \;\leq\; \pi/4,$$

then the term 1 in the

$$\tau^{-1} \;=\; \frac{1}{\sigma(1 + 2\pi i\sigma^{-1}t)} \tag{3}$$

is dominant, and we get that

$$\tau^{-1} \;=\; \frac{1}{\sigma(1 + 2\pi it\sigma^{-1})} \;=\; \sum_{j=0}^{\infty} \sigma^{-1-j}(-1)^j(2\pi it)^j$$

converges nicely. We really only need the real part of this series to determine an upper bound on $|f(\tau)|$; and so, we need only consider

$$\mathrm{Re}(\tau^{-1}) \;=\; \sum_{k=0}^{\infty}(-1)^k\sigma^{-1-2k}(2\pi t)^{2k}.$$

For our purposes we only need to consider the first two terms here, which provide the nice estimate

$$\mathrm{Re}(\tau^{-1}) \;<\; \sigma^{-1} - c\sigma^{-1/5}, \qquad (4)$$

for some $c > 0$.

On the other hand, if $\sigma/10 \leq |t| \leq 1/2$, then the term $2\pi i t\sigma^{-1}$ in (3) is of comparable size to the term 1; so, it is not difficult to see that this implies

$$|\tau^{-1}| \;<\; (1-c)\sigma^{-1}. \qquad (5)$$

(We can use the same $c > 0$ as in (4), if we just take it small enough.)

3.1.2 The easy case: $t > \sigma^{7/5}$, $|\mathrm{Arg}(\tau)| > \pi/4$

For this case, we just use the second part of Theorem 1, and we get a nice upper bound for $|f(\tau)|$.

3.1.3 Completion of the estimate for E

What we deduce from (4), (5), and Theorem 1 is that if $|t| > \sigma^{7/5}$, then

$$|f(\tau)| \;<\; f(\sigma)\exp(-C\sigma^{-1/5}), \text{ for some } C > 0.$$

Using this, we easily obtain the bound

$$|E| \;<\; e^{n\sigma}f(\sigma)\exp(-C\sigma^{-1/5}).$$

One sees that this is much much smaller than the value of $p(n)$ claimed by our Theorem 2, so this error term is quite small indeed (in fact smaller than $O(p(n)n^{-1/10})$!

3.2 Estimation of the "main term"

Let us now therefore focus on the "main term", which is just

$$e^{n\sigma}\int_{-\sigma^{7/5}}^{\sigma^{7/5}} f(\sigma+2\pi it)e^{2\pi int}dt.$$

Since all the values of $\tau=\sigma+2\pi it$ satisfy $|\mathrm{Arg}(\tau)|\leq\pi/4$ for this integral, the first part of Theorem 1 applies. In order to work out what Theorem 1 gives, we observe that for $|t|\leq\sigma^{7/5}$,

$$\tau^{-1} \;=\; \frac{1}{\sigma(1+2\pi it\sigma^{-1})} \;=\; \frac{1}{\sigma} \;-\; \frac{2\pi it}{\sigma^2} \;-\; \frac{4\pi^2t^2}{\sigma^3} \;+\; O(\sigma^{1/5}). \qquad (6)$$

3.2.1 Commentary on our choice $|t|\leq\sigma^{7/5}$

Now we see where our chose of $|t|\leq\sigma^{7/5}$ comes from:

First, if we replaced the exponent 7/5 with any number 4/3 or smaller, we would not have gotten that the fourth term on the right-hand-side of (6) is as small as $O(\sigma^{1/5})$; in fact, with exponent 4/3 the corrsponding term would have been only $O(1)$.

Second, if we had replaced the 7/5 with any number exceeding 3/2, it is easy to see we would not have gotten the estimate (4).

A third place where our choice of exponent 7/5 comes up near the end of our proof, where a certain integral comes up that is only well-approximable by a Gaussian integral when the our exponent is smaller than 3/2.

3.2.2 Conclusion of the main term estimate

Next, we consider the term $\log(\tau)$: We have that

$$\begin{aligned}\log(\tau) &= \log|\tau|+i\mathrm{Arg}(\tau)\\ &= \log|\sigma(1+2\pi it\sigma^{-1})|+O(\sigma^{2/5})\\ &= \log(\sigma)\;+\;O(\sigma^{2/5}).\end{aligned}$$

Putting everything together, we find that for $|t| \le \sigma^{7/5}$,

$$f(\tau)e^{2\pi int} \;=\; \frac{(1+O(\sigma^{1/5}))\sqrt{\sigma}}{\sqrt{2\pi}}\exp\left(\zeta(2)\left(\frac{1}{\sigma}-\frac{2\pi it}{\sigma^2}-\frac{4\pi^2t^2}{\sigma^3}\right)+2\pi int\right).$$

We get some simplification arising from the fact that $\sigma = \sqrt{\zeta(2)/n}$; in fact, the two terms involving $2\pi it$ will cancel to give

$$f(\tau)e^{2\pi int} \;=\; \frac{(1+O(\sigma^{1/5}))\sqrt{\sigma}}{\sqrt{2\pi}}\exp\left(\frac{\zeta(2)}{\sigma}-\frac{4\zeta(2)\pi^2t^2}{\sigma^3}\right).$$

Multiplying this by $e^{n\sigma}$ and integrating over $|t| \le \sigma^{7/5}$ we find that

$$\begin{aligned} p(n) \;&=\; \frac{(1+O(\sigma^{1/5}))\sqrt{\sigma}}{\sqrt{2\pi}}\exp\left(\frac{\zeta(2)}{\sigma}+n\sigma\right)\int_{-\sigma^{7/5}}^{\sigma^{7/5}}\exp\left(-\frac{4\zeta(2)\pi^2t^2}{\sigma^3}\right)dt \\ &=\; \frac{(1+O(\sigma^{1/5}))\kappa\sqrt{\sigma}}{\sqrt{2\pi}}\exp\left(2\sqrt{n\zeta(2)}\right)\int_{-\kappa^{-1}\sigma^{7/5}}^{\kappa^{-1}\sigma^{7/5}}\exp\left(-u^2/2\right)du \\ &=\; \frac{(1+O(\sigma^{1/5}))}{4n\sqrt{6\pi}}\exp\left(2\sqrt{n\zeta(2)}\right)\int_{-\kappa^{-1}\sigma^{7/5}}^{\kappa^{-1}\sigma^{7/5}}\exp\left(-u^2/2\right), \end{aligned}$$

where

$$t \;=\; \kappa u, \quad \text{and} \quad \kappa = (\sigma/2\pi)\sqrt{\sigma/2\zeta(2)}.$$

Since $\kappa^{-1}\sigma^{7/5}$ grows like a constant times $\sigma^{-1/5}$, we find that the above integral can be very well approximated (to at least an error as good as $O(\sigma^{1/5})$, and in fact much better) by extending the range of integration to infinity. When we do this, our integral is well-known to have value $\sqrt{2\pi}$; and so, we find that

$$p(n) \;=\; \frac{(1+O(n^{-1/10}))}{4n\sqrt{3}}\exp\left(\pi\sqrt{2n/3}\right),$$

as claimed.

4 Proof of Theorem 1

We first observe that we have no hope of saying anything intelligent about f when we write it as a series (i.e. in terms of $p(n)e^{-n\tau}$); and so, we have to work with the nicer, product expansion, and an obvious way to begin is to take logs. When we do this, and interchange sums (which are possible since $\sigma > 0$) we obtain the deductions

$$\log f(\tau) \;=\; \sum_{k\geq 1}\sum_{j\geq 1}\frac{e^{-kj\tau}}{j} \;=\; \sum_{j\geq 1}\frac{1}{j}\sum_{k\geq 1}e^{-kj\tau}. \tag{7}$$

Now what do we do? Well, the inner sum is a geometric series, but after some work and thought, one will see that one cannot (easily) get anywhere by using the geometric series formula. Another idea is to transform the series into a zeta series by taking the Mellin transform of both sides. The trouble with that idea is that we would then have to invert the whole thing to find $\log f(\tau)$, which could be messy. Meinardus's idea was to directly write the $e^{-kj\tau}$ in terms of a contour integral involving the Γ-function and the Riemann ζ-function, and the way he did this was through the "inverse Mellin transform".

4.1 Mellin and inverse Mellin transforms

Let us briefly recall some facts about the Mellin transform: Given a function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$, the "Mellin transform" of f is defined to be

$$\mathcal{M}(f) \;:=\; \int_0^\infty f(t)t^{s-1}dt,$$

provided, of course, that the integral converges. It typically takes a function f defined on the reals, and produces a function that is analytic in some region of $\mathbb{C}$. It is a basic sort of integral transform, much like, for example, the Laplace transform.

The inverse Mellin transform, on the other hand, takes a function $g(s)$ defined in some region of the complex plane of the form

$$c - \epsilon \;<\; \mathrm{Re}(s) \;<\; c + \epsilon,$$

and produces a real-valued function $f(t)$ via the transform

$$\mathcal{M}^{-1}(g) \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)t^{-s}ds.$$

As we saw in class when we worked with theta functions,

$$e^{-t} \xrightarrow{\mathcal{M}} \Gamma(s).$$

So,

$$e^{-t} \xleftarrow{\mathcal{M}^{-1}} \Gamma(s).$$

That is, for $c > 0$,

$$e^{-t} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) t^{-s} ds. \tag{8}$$

As we will see in a later subsection, $\Gamma(s)$ decays very rapidly to 0 if its real part is held fixed, but imaginary part tends to infinity; so, this integral certainly conveges.

4.2 A formula for $\log f(\tau)$

Plugging in (8) to (7) we find that for any $c > 0$,

$$\log f(\tau) = \sum_{j\geq 1} \frac{1}{j} \sum_{k\geq 1} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)(kj\tau)^{-s} ds.$$

If we assume that $\mathrm{Re}(s) > 1$, then it is easy to show that we can interchange the integral and sums, because the sum over k and j is absolutely convergent, and because the $\Gamma(s)$ decays rapidly to 0 as we integrate up the line $c + it$. When we do this, and in fact when we let $c = 2$ (which certainly means $\mathrm{Re}(s) = c > 1$ for those s arising in our integral), we find that

$$\begin{aligned} \log f(\tau) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \tau^{-s}\Gamma(s)\left(\sum_{j\geq 1}\frac{1}{j^{s+1}}\right)\left(\sum_{k\geq 1}\frac{1}{k^s}\right) ds \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1) ds. \end{aligned} \tag{9}$$

The idea is to now express this integral as a finite integral over some box, in order to pick up the residues at poles of $\zeta(s)\zeta(s+1)\Gamma(s)$ at $s = 0$ and $s = 1$. In order to do this, we will need to know a few things about $\zeta(s)$ and $\Gamma(s)$ (actually, we have already used some of the facts about these functions that we will prove below!).

4.3 Some facts about $\zeta(s)$ and $\Gamma(s)$

In order to exploit the Mellin-inversion formula from the previous subsection, we need to keep in mind a few facts about $\zeta(s)$ and $\Gamma(s)$.

First, a very crude deduction from Stirling's formula is that for every $\epsilon > 0$ if

$$-\pi + \epsilon \ < \ \mathrm{Arg}(s) \ < \ \pi - \epsilon,$$

then

$$\log \Gamma(s) \ = \ s \log s \ - \ s + o_\epsilon(s).$$

So, if we fix σ and let $s = \sigma + it$, where $t \to \infty$, then we find that

$$\log \Gamma(s) \ \sim \ (\sigma + it)(\log t + i\pi/2) + o(t) \ = \ -t\pi/2 + i(t \log t) + o(t).$$

So,

$$|\Gamma(s)| \ = \ \exp((-\pi/2 + o(1))t).$$

As $\Gamma(s) = \overline{\Gamma(\overline{s})}$, we deduce a similar result for when $t \to -\infty$, and putting both cases ($t \to \infty$ and $t \to -\infty$) together, we find that

Lemma 1 *For $\sigma \in \mathbb{R}$ fixed, if $s = \sigma + it$, then as $t \to \infty$ or $-\infty$ we have that*

$$|\Gamma(s)| \ = \ \exp((-\pi/2 + o(1))|t|).$$

Now we show a similar sort of thing about $\zeta(s)$. First, note that from the analytic continuation formula

$$\zeta(s) \ = \ \frac{s}{s-1} \ - \ s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx, \ \mathrm{Re}(s) > 0, \ s \neq 1,$$

one can easily show the following:

Lemma 2 *If $\sigma_2 > \sigma_1 > 0$, and $s = \sigma + it$ satisfies*

$$\sigma_1 \ < \ \sigma \ < \ \sigma_2, \text{ and } |s - 1| \ > \ \epsilon,$$

then

$$|\zeta(s)| \ < \ h(\epsilon, \sigma_1),$$

for some function $h \ : \ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$.

A routine application of the functional equation for $\zeta(s)$, along with bounds for $\Gamma(s)$ we developed in Lemma 1 allows us to deduce a similar result for $\sigma < 0$, and putting together both the $\sigma > 0$ and $\sigma < 0$ cases we deduce

Lemma 3 *Given $\sigma_2 > \sigma_1$, there exists $\delta > 0$ so that: If $s = \sigma + it$ satisfies*

$$\sigma_1 \; < \; \sigma \; < \; \sigma_2, \text{ and } |s-1| \; > \; \epsilon,$$

then

$$|\zeta(s)| \; = \; O((|s|+2)^{\delta}).$$

Finally, we will require the following, which we will not bother to prove – the proofs are consequences of Stirling's formula and the functional equation of $\zeta(s)$:

Lemma 4 *We have that*

- $\zeta(0) = -1/2$.
- $\zeta'(0) = -\log(2\pi)/2$.
- *About $s = 1$ we have the Laurent expansion*

$$\zeta(s) \; = \; \frac{1}{s-1} \; + \; \gamma \; + \; \text{higher order terms},$$

where γ is Euler's constant.

- *About $s = 0$ we have the Laurent expansion*

$$\Gamma(s) \; = \; \frac{1}{s} \; - \; \gamma \; + \; \text{higher order terms}.$$

4.4 Resumption of our estimate for $f(\tau)$.

We note that if we trucate the integral in (9) at height T, where we think of T as "near infinity", so that we are considering the integral

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1)ds,$$

we get a very good approximation, because by Lemma 1 above we find that the $|\Gamma(s)| = \exp((-\pi/2 + o(1))|t|)$ for $s = 2 + it$, and the factor $\zeta(s)\zeta(s+1)$ is bounded along this line $2 + it$.

In order to relate this integral to properties of ζ and Γ, we complete it to a box, and show that the three new sides have negligable contribution to the integral, at least for certain values of τ: Suppose that our box has corners

$$2 - iT, \quad 2 + iT, \quad -1/2 + iT, \quad -1/2 - iT.$$

(In place of the $-1/2$ we could use any number in $(-1, 0)$ to get a good estimate – the reason for using this interval is so that we pick up a residue at $s = 0$ from various factors.)

4.4.1 The case $|\mathrm{Arg}(\tau)| \leq \pi/4$

Let us first work with the case where

$$|\mathrm{Arg}(\tau)| \ \leq \ \pi/4.$$

Consider the two segments $2+iT \to -1/2+iT$ and $-1/2-iT \to 2-iT$. The integral over both of these segments dies away to 0 as $T \to \infty$, since the ζ factors have at most polynomial growth in $|s|$, and the Γ factor dies away to 0 exponentially fast.

Next, consider the segment $-1/2 + iT \to -1/2 - iT$. In order to understand how the integral behaves along this segment, first suppose that

$$|\mathrm{Arg}(\tau)| \ \leq \ \pi/4.$$

Then, we have that $\tau = \sigma + 2\pi i t$ has the property that

$$|\tau| \ \leq \ \sqrt{2}\sigma,$$

which will allow us to express certain errors in terms of σ. First, we realize that along the line with $\mathrm{Re}(s) = -1/2$,

$$\begin{aligned} |\tau^{-s}| \ = \ |\exp(-s\log\tau)| \ &= \ \exp(1/2\log|\tau| + \mathrm{Im}(s)\mathrm{Arg}(\tau)) \\ &\leq \ |\tau|^{1/2}\exp(\pi|\mathrm{Im}(s)|/4). \end{aligned}$$

As bad as this exponential factor looks, it does not grow to infinity as fast as the Γ factor tends to 0, because of the $\pi/2$ appearing in Lemma 1. So, we have that

$$\left| \frac{1}{2\pi i} \int_{-1/2-iT}^{-1/2+iT} \tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1)ds \right| \;=\; O(|\tau|^{1/2}) \;=\; O(\sigma^{1/2}).$$

Putting everything together, we know that by choosing T large enough, we will have that

$$\log f(\tau) \;=\; \mathrm{Res}_{s=0}\tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1) + \mathrm{Res}_{s=1}\tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1) + O(\sigma^{1/2}).$$

The residue at $s = 1$ is easy to compute, but the one at $s = 0$ is much trickier, because there is a pole of order 2 at $s = 0$. First,

$$\mathrm{Res}_{s=1}\tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1) \;=\; \tau^{-1}\zeta(2)\mathrm{Res}_{s=1}\zeta(s) \;=\; \tau^{-1}\zeta(2).$$

For the residue at $s = 0$, we use Lemma 4, along with the fact that

$$\tau^{-s} \;=\; e^{-s\log\tau} \;=\; 1 - s\log\tau + O(s^2), \text{ and } \zeta(s) \;=\; \zeta(0) + s\zeta'(0) + O(s^3),$$

and deduce that about $s = 0$ the prduct $\tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1)$ has the Laurent expansion

$$\begin{aligned}&(1 - s\log\tau + \cdots)(1/s - \gamma + \cdots)(1/s + \gamma + \cdots)(\zeta(0) + s\zeta'(0) + \cdots)\\ &\quad= \zeta(0)/s^2 + (\zeta'(0) - \zeta(0)\log\tau)/s + \cdots .\end{aligned}$$

From this and Lemma 4 we deduce that

$$\begin{aligned}\mathrm{Res}_{s=0}\tau^{-s}\Gamma(s)\zeta(s)\zeta(s+1) \quad &= \quad \zeta'(0) - \zeta(0)\log\tau\\ &= \quad -\log(2\pi)/2 + \log(\tau)/2.\end{aligned}$$

It follows that

$$\log f(\tau) \;=\; \tau^{-1}\zeta(2) - \log(2\pi)/2 + \log(\tau)/2 + O(\sigma^{1/2}). \qquad (10)$$

Exponentiating, we have the first part of our theorem!

4.4.2 The case $\pi/4 < |\mathrm{Arg}(\tau)| \leq \pi/2$

Next, we consider our integral for the case when

$$\pi/4 \ < \ |\mathrm{Arg}(\tau)| \ \leq \ \pi/2.$$

For this case all we will neeed are rough upper bound on $f(\tau)$, because in our saddlepoint method approach these τ will occur only in our "error term" estimates.

The way to handle this case is to relate $\log f(\tau)$ to $\log f(\sigma)$ using a trick. Before we explain the trick, we define

$$g(\tau) \ = \ \sum_{n=1}^{\infty} e^{-n\tau} \ = \ \frac{e^{-\tau}}{1-e^{-\tau}},$$

by the geometric series identity. Then, we observe that

$$\begin{aligned} \mathrm{Re}(\log f(\tau) - g(\tau)) \quad &= \quad \sum_{j=2}^{\infty} \frac{1}{j} \sum_{n=1}^{\infty} \mathrm{Re}(e^{-jn\tau}) \\ &\leq \quad \sum_{j=2}^{\infty} \frac{1}{j} \sum_{n=1}^{\infty} e^{-jn\sigma} \\ &= \quad \log f(\sigma) - g(\sigma). \end{aligned} \tag{11}$$

So,

$$\mathrm{Re}(\log f(\tau)) \ \leq \ \log f(\sigma) - g(\sigma) + \mathrm{Re}(g(\tau)).$$

Since $\mathrm{Arg}(\sigma) = 0$, we can use our previous estimates for $f(\tau)$ on the term $f(\sigma)$; further, since g is just a geometric series, the terms $g(\sigma)$ and $g(\tau)$ can be easily computed. Let us first consider there terms $g(\sigma)$ and $g(\tau)$: Observe that

$$\begin{aligned} |1-e^{-\tau}| \quad &= \quad |(1-e^{-\sigma}\cos(2\pi t)) + i\sin(2\pi t)| \\ &\geq \quad |(1-e^{-\sigma}) + i\sin(2\pi t)|. \end{aligned}$$

We now consider two cases: Case 1 is when $|t| < 1/4$, and case 2 is when $1/4 \leq |t| \leq 1/2$. In either case we, of course, also assume that $|\mathrm{Arg}(\tau)| > \pi/4$.

Case 1 ($|t| < 1/4$).

For this case we start with the inequalities

$$1 - e^{-\sigma} \ < \ 1 - e^{-2\pi|t|} \ < \ \sin(2\pi|t|). \tag{12}$$

(This right-most inequality is easily proved via Taylor expansions.) What this means is that the imaginary component of $1 - e^{-\tau}$ is at least of comparable size to its real component; and therefore, we will have for σ near 0 (which is certainly is)

$$|1 - e^{-\tau}| \ \gtrsim \ \sqrt{(1 - e^{-\sigma})^2 + \sin^2(2\pi t)} \ > \ \sqrt{2(1 - e^{-\sigma})^2} \ = \ \sqrt{2}(1 - e^{-\sigma}).$$

It now follows that for σ near 0,

$$\begin{aligned} \mathrm{Re}(g(\tau)) - g(\sigma) \ &\leq \ \frac{e^{-\sigma}}{\sqrt{2}(1 - e^{-\sigma})} - \frac{e^{-\sigma}}{1 - e^{-\sigma}} \\ &\leq \ -\frac{(1 - 1/\sqrt{2})e^{-\sigma}}{(1 - e^{-\sigma})} \\ &\lesssim \ -\frac{1 - 1/\sqrt{2}}{1 - e^{-\sigma}} \\ &< \ -\frac{1 - 1/\sqrt{2}}{\sigma}. \end{aligned}$$

Combining this with (10) and exponentiating, we deduce that

$$|f(\tau)| \ \leq \ f(\sigma)\exp(-(1 - 1/\sqrt{2} + o(1))\sigma^{-1}).$$

Case 2 ($1/4 \leq |t| \leq 1/2$).

The reason we have to treat this case seperately is that (12) fails to hold when $|t|$ is near $1/2$. Fortunately, there is something else we can use to work around this problem: We have that

$$\mathrm{Re}(1 - e^{-\tau}) \ = \ 1 - e^{-\sigma}\cos(2\pi t) \ \geq \ 1.$$

So, for $\sigma < 1/6$ we will have

$$\mathrm{Re}(g(\tau)) - g(\sigma) \ \leq \ 1 \ - \ \frac{1}{e^{\sigma} - 1} \ < \ 1 - \frac{1}{2\sigma} \ < \ -\frac{1}{3\sigma}.$$

So, in this case we will also get that

$$|f(\tau)| \ \leq \ f(\sigma)\exp(-(1/3+o(1))\sigma^{-1}).$$

Our Theorem is now proved! ■